Epistemological Demands of the Second Discontinuity

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Mathematics at university presents highly condensed knowledge which is based on set-theoretic definitions and formal proof and is largely freed from contexts. This is the final stage of a long-term development of mathematical thinking which begins with physical perception and action in concrete situations and grows in a long process of reformulation and sophistication. When a young student returns from university to school as a teacher, he or she must return to the roots and accompany and support his or her students in this process. This task contains multiple epistemological requirements which will be addressed in the following paper.

Keywords: second discontinuity, stages of knowledge building, epistemological awareness, epistemological obstacles, concept changes

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Les exigences épistémologiques de la deuxième discontinuité

Les mathématiques à l'université présentent des connaissances très condensées qui sont basées sur des définitions de la théorie des ensembles et des preuves formelles et qui sont en grande partie libérées des contextes. Il s'agit de la dernière étape d'un développement à long terme de la pensée mathématique qui commence par la perception physique et l'action dans des situations concrètes et se développe dans un long processus de reformulation et de sophistication. Lorsqu'un jeune étudiant revient de l'université à l'école en tant qu'enseignant, il doit revenir aux sources et accompagner et soutenir ses élèves dans ce processus. Cette tâche comporte de multiples exigences épistémologiques qui seront abordées dans le document suivant.

Mot-clés: deuxième discontinuité, étapes de la construction des connaissances, conscience épistémologique, obstacles épistémologiques, changements de concepts

Exigencias epistemológicas de la segunda discontinuidad

Las matemáticas en la universidad presentan conocimientos muy condensados que se basan en definiciones teóricas de conjuntos y en pruebas formales y están en gran medida liberadas de contextos. Se trata de la etapa final de un desarrollo a largo plazo del pensamiento matemático que comienza con la percepción física y la acción en situaciones concretas y crece en un largo proceso de reformulación y sofisticación. Cuando un joven estudiante vuelve de la universidad a la escuela como profesor, debe volver a las raíces y acompañar y apoyar a sus alumnos en este proceso. Esta tarea contiene múltiples exigencias epistemológicas que se abordarán en el siguiente trabajo.

Palabras claves: segunda discontinuidad, etapas de la construcción del conocimiento, conciencia epistemológica, obstáculos epistemológicos, cambios de concepto

1. Introduction

The following reflections are concerned with the second discontinuity in the sense of Felix Klein (Klein, 2016, p. xiii), which refers to the transition from academic studies to working as a mathematics teacher. They address difficulties that secondary school teachers face when they are confronted with a completely different level of knowledge formation than they were used to at university.

Mathematics at university is based on set-theoretic definitions and formal proof and is therefore largely freed from particular contexts. This is the final stage of a long-term development of mathematical thinking which begins with physical perception and action and grows in a long process of reformulation and sophistication (Tall, 2013). When a young student returns to school as a teacher, he or she must return to the roots and accompany and support the students in this process.

This task requires content knowledge adapted to the disposition of the learners and sensitivity for subject-related learning processes. Regarding the learning content, previously acquired knowledge of school mathematics can be activated and enriched by suggestions from the literature (teaching materials, further training materials, relevant didactic publications). The second area includes an awareness of the ways in which mathematical thinking develops, which stages are passed through and which difficulties must be overcome. The considerations in this paper focus on these epistemological challenges. Their horizon can be outlined by the following statement:

> The long-term development of mathematical thinking is [...] more subtle than adding new experiences to a fixed knowledge structure. It is a continual reconstruction of mental connections that evolve to build increasingly sophisticated knowledge structures over time. (Tall, 2013, p. 6).

It is not possible to deal with this task exhaustively here. For such a purpose, the entire problem-related state of research would have to be reviewed. I will therefore concentrate on three problem areas that I experienced as particularly sensitive as a young teacher and that I have pursued throughout my professional life. The considerations are naturally influenced by the underlying teaching philosophy. The aim is a teaching style that is oriented towards the natural conditions of professional knowledge development and endeavors to bring out the fundamental characteristics of mathematical thinking in an elementary context.

Academic mathematical conceptual knowledge is based on general concepts with a wide range of applications. These include an elaborate concept of numbers, a broad concept of functions and algebraic structural concepts such as groups, fields and vector spaces. A genetically oriented mathematics education must start with phenomena which can be organized by such concepts and enrich the understanding by phenomena to which these concepts can been extended (Freudenthal, 1983). Academic mathematical operational knowledge, e.g. the handling of symbol systems, is often based on subconsciously practiced familiarities whose internal laws must be brought to light again for teaching purposes. Both requirements will be summarized under the title "Return of condensed knowledge to a high resolution" and dealt with in section 2.

The formation of "increasingly sophisticated knowledge structures over time" (see above) requires a sensitive awareness of stages of knowledge formation with respect to suitable starting points, driving questions, accompanying forms of expression in language and representations and specific forms of reassurance. This problem area is discussed in section 3 under the heading "Awareness of stages of knowledge formation".

Further, for cognitively activating lessons, it is important to identify and bring to bear the driving questions of mathematical knowledge building in an elementary context. This requirement is the subject of section 4.

The aspects presented in these sections are not independent of each other. The classification was made according to typical focal points.

In a conclusion in section 5 a summarizing consideration is given to how a suitable preparation of content knowledge could support teachers in meeting these requirements.

2. Return from condensed knowledge to a high resolution

In a practical training at school a trainee treated applications of the greatest common divisor (gcd) and least common multiple (lcm) in a 6th grade class. She set the following task:

> A garden 168 dm long and 96 dm wide is to be fenced. On each side the posts should have equal distance. How big do you have to choose this distance if you want to use as few posts as possible?

It was one of the typical tasks that pretend to be realistic but are only tailored to the application of a specific calculation method – however. The students in the last row started to discuss lively. From the classroom context, they concluded: greatest common divisor or least common multiple, that is the question here. "You can't take the least common multiple, you only have this circumference!" one student pointed out. With unerring intuition, he connected the gcd with limited resources and the lcm rather with open processes. Thus, the selection of possible answers was narrowed down based on situational plausibility. The trainee took up the suggestion "gcd", it brought her onto familiar terrain. The calculation on the blackboard could begin.

> $168 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7$
> $96 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$
> $\gcd(168,96) = 2 \cdot 2 \cdot 2 \cdot 3 = 24$
> The distance between the posts is 24 dm.

Finally, a student timidly asked, "How wide are the posts, anyway?" This question faded away unused. But it revealed an original point of view:

1. *Why is it possible to neglect the width of the posts in this situation?*
2. *To what extent is the achieved result honest?*

What awareness does a trainee need to develop for situations like this? Mathematical model assumptions are not self-explanatory. You need to understand a situation well to make reasonable assumptions or to understand assumptions that have been made. And the result should be checked again against reality: Does the width of the posts not matter here? In what sense are the idealizations made permissible, and the calculations performed useful? Presumably, the students should have used sketches and perhaps practical measurements to get a more accurate picture of the situation. Without these experiences, they learned no more than a school ritual.

In his famous book "Didactical Phenomenology of Mathematical Structures" Freudenthal (1983) develops the following philosophy:

> Our mathematical concepts, structures, ideas have been invented as tools to organize the phenomena of the physical, social, and mental world. *Phenomenology* of a mathematical concept, structure, or idea means describing it in its relation to the phenomena for which it was created, and to which it has been extended in the learning process of mankind, and, as far as this description is concerned with the learning process of the young generation, it is *didactical phenomenology*, a way to show the teacher the places where the learner might step into the learning process of mankind. Not in its history but in its learning process that still continues, which means that dead ends must be cut and living roots spared and reinforced. (Freudenthal, 1983, Preface, p. ix)

For the learners in the episode described above, the way in which the mathematical structure used describes the—in this case non-mathematical—situation should have been examined more closely. For this purpose, the trainee would have had to scrutinize thought processes that had become habitual and trace them back to their origins, and that means to see the connections in a higher resolution. This requirement is considered in more detail using selected examples in the following two subsections for mathematical concepts and procedures.

2.1 Mathematical concepts

It is very quiet. A fifth-grade class is writing a geometry test on the topic of "sphere". The teacher has asked the question: "Are all circles of latitude equally long?"[1] *Anja writes down: "The circles of latitude are not all equally long. By the way, they are not long, but round and around the earth.*

1. Perhaps "the same length" would be better English, but the chosen translation corresponds exactly to the German original, and that is important for the interpretation here.

> *Each circle of latitude has a partner on the opposite side."(Andelfinger, 1990, p. 57, translated by the author).*

Anja seems alert enough to reproduce the expected answer and confident enough, to make her reservations explicit. She seems to associate "long" with "straight" and perhaps "far". She must experience that this view can be extended and that you can also measure curved figures with a tape measure and thus develop a more sophisticated meaning of the adjective "long". So, she will finally understand the term "length" as a comprehensive concept and thus be prepared to determine the circumference of a circle and arc lengths using infinitesimal methods at a later stage.

Perhaps it is no coincidence that Freudenthal (1983) begins his phenomenology with the concept of length and develops features of an extended learning process from the first early experiences with length comparison and length measurement ("Are the sleeves long enough?") to the mathematical calculation of curve lengths. His explanations show a high level of awareness of the diversity of individual phenomena that ultimately can be subsumed under a common concept. The variety of descriptive adjectives is correspondingly large: "long, short, tall, broad, tight, high, low, deep, shallow, far, near, wide, narrow" etc. "For the adult it is—at least unconsciously—clear how these expressions are related to the same magnitude, length, and he often presupposes children to be well acquainted with this relation." (ibid, p. 11). An awareness of this phenomenological variety and an understanding of the associated difficulties must be gained or regained for sensitive teaching.

The concept of fractions is also multifaceted and very difficult due to its many different manifestations. (Freudenthal, 1983, Chapter 5):

- Fraction as a proportion: this fractional aspect is directly related to the verb "to break" (the Latin noun is fractio): To be able to divide fairly, stocks are broken into pieces of the same size.
- Fraction as an instrument of comparison: This usage occurs in statements about size comparison: "Rod A is 5/6 times as long as rod B." or: "There are 2 1/2 times as many adults in the room as children." Nothing is "broken into pieces", but objects are placed in relation with respect to their size. Here, fractions have the meaning of "ratio" as an expression of numerical relationships. This is the linguistic root of the technical term "rational number".
- Fraction as operator: Statements such as "5/6 of" or "2 1/2 times as much" can be viewed from a higher-level perspective as the application of an operator to a unit. This leads to an understanding that encompasses many manifestations.
- Fraction as a magnitude: Statements such as 1/2 m, 1/8 kg or 3/4 h can be subsumed under the operator aspect or viewed as an independent number aspect.

But there are also subtle differences within one and the same aspect. Forming or recognizing a proportion of a compact object (example: cake) or a discrete set of objects (example: string of pearls) can pose different challenges for learners. Here too, teachers must learn to see the landscape of embodiments of the fraction concept in a high resolution and realize that their mathematical similarities are not self-evident from the very beginning.

2.2 Mathematical operations

> *While doing her homework, Katharina had correctly applied the rules for computing fractions and divided the number 2 by ¼. Her result, 8, surprised her so much that she then asked me how the result could possibly be larger than the dividend. After all, she had 'divided' it! I attempted to explain to her why this must be the case (for positive numbers) with division by numbers smaller than 1. Finally, she remarked, "Okay. Now I know how to compute this. But don't try to tell me that math has to do with logical thinking." (Heymann, 2003, p. 156)*

Katharina obviously gets into conflict with habits that she has formed when calculating with natural numbers. Basic ideas about the calculation methods for natural numbers have their roots in elementary action situations:

- Addition: joining, adding, multiplying.
- Subtraction: taking away, decreasing.
- Multiplication: taking and adding the same amount several times, measuring the overall effect.
- Division: dividing (fairly), measuring out with equal parts, several times.

These basic ideas of arithmetic operations as well as connotations of their accompanying effects must be appropriately expanded or even modified at each new level in the expansion of the number system. In fractions, for example, multiplication must be extended by the idea of forming a proportion. Division by a fraction can no longer be explained as distributing to a whole number of addressees, while the idea of measuring out remains valid and can explain calculations such as 4:1/2=8. In more complicated cases ("How often do 2/5 fit into 3/4?"), an interpretation based on visualization becomes challenging as well as the proportion of a proportion.

Sensitive teaching here involves the requirement to break down an acquired superordinate number concept back into its manifestations, to consider each of these individually and in their interconnectedness and to become sensitive to all the comprehension problems that an extended understanding of numbers and operations with numbers entails.

These problems begin with the written calculation methods for natural numbers, they continue in arithmetic with fractions and negative numbers and above all with symbolic arithmetic in elementary algebra. In general, it can be said that the automation of operations which experts take for granted requires many learning steps, for the sequence of which and the accompanying learning hurdles teachers must develop a new awareness.

3. Awareness of stages of knowledge formation

As a part of a practical training for my students of teaching profession, I delivered a lesson to a fifth-grade high school class on the following task from a textbook (Czech, 1989):

> In the land of Fantasia, the cities are connected only by one-way streets. Fares in FM (Fantasia mark) are indicated in the boxes. We want to go from city A to city B.
>
> a) How much does it cost to go via d, e, f?
> b) What is the cheapest way?
> c) Is there a way that costs 38 FM?
> d) What is the most expensive way?

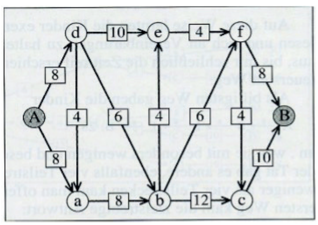


Figure 1. –Travel costs in Fantasia

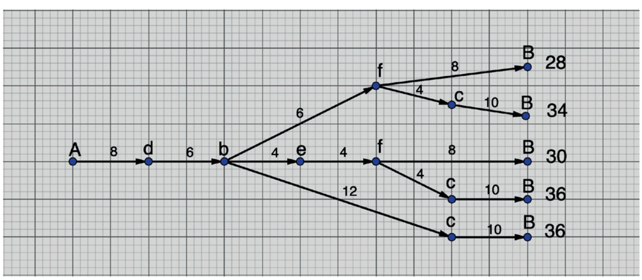


Figure 2. – Section of a tree diagram[2]

The students liked the task, worked intensively, and performed well. They answered questions b) and d) with intuitive certainty. In the next lesson, the class teacher created a tree diagram of all possible ways from A to B using a prepared worksheet. However, the process was tough, and it turned out that the context of the representation appeared very foreign to the students.

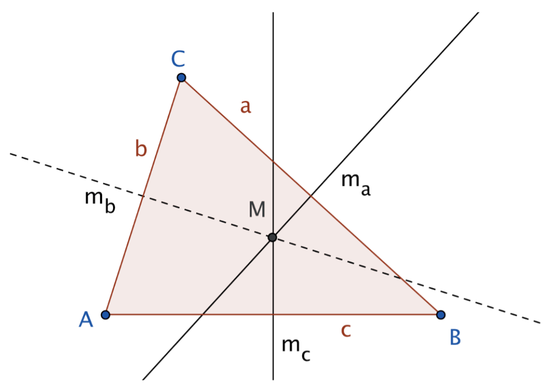
2. As the complete tree diagram is extensive, a section has been selected here. It shows all paths that begin with the route sequence A→d→b.

Apparently, there was a shift in the style of knowledge building between these two lessons. Although the initial situation was artificial, it was based on a real situation. In the first lesson, the solutions were obtained by direct calculation; certainty resulted from empirical plausibility and the answers were good enough for the purpose in hand. The second lesson had a different objective. It was about analytically secured knowledge based on a systematically gained overview by methods with a scope of application that goes beyond the specific situation. The presentations differed accordingly. The route map (Figure 1) reminded of road maps and in this respect had an iconic character in the sense of Peirce (Hoffmann, 2005), the tree diagram (Figure 2) was an analytical tool that no longer bore any resemblance to a real situation and required new interpretative skills to be developed. It is possible that the second type of question and the means of investigating it were used too early in this age group so that the learners did not see the necessity and did not share the triggering interest. But this is a question that needs further clarification by empirical research.

What happens here within a localized topic extends over longer periods of time in the case of comprehensive subject areas. The following two subsections take a closer look at two important areas of school mathematics, geometry, and arithmetic/algebra. The study thus remains exemplary. An extension to further areas, in particular functions/analysis and stochastics, would be desirable but is beyond the scope of this article.

3.1 Stages of knowledge formation in geometry

It happened during a school practicum in a middle school class, where a trainee dealt with the theorem of the intersection of the perpendicular bisectors in a triangle. He developed a drawing on the blackboard (Fig. 3) and supporting arguments for a proof in the discussion.



Proof:
Let M be the intersection point of m_a and m_c. Since M lies on m_a, M is equidistant from B and C. Since M lies on m_c, M is equidistant from A and B. Then M is also equidistant from A and C. So, M must also lie on m_b. This means that all three perpendiculars pass through M.

Figure 3. – Intersection of the perpendicular bisectors in a triangle

> *Finally, the trainee summarized the result of the effort: "Now we have shown that the perpendicular bisectors in the triangle intersect at a point." He looked very satisfied with this insight. Within an informal communication, one student in the last row quietly said to his neighbor: "That they intersect at a point, you can see this. What is interesting is whether they also intersect in the middle of the triangle." And another said: "Now we see it for this triangle. How about other ones?"*

It is often the informal communication in the classroom that provides information about fundamental problems of understanding. The learners and the trainee obviously differed fundamentally in their assessment of this geometric situation with respect to what was interesting and worth investigating, what could be taken for granted and what required justification. They were therefore at different stages of geometric sophistication.

As Struve (1991) has shown in a very revealing analysis (cf. Houdement & Kuzniak, 1999), geometry is practiced as empirical geometry at the beginning of secondary school (grades 5 and 6). Its terms serve as a means of describing and explaining empirical phenomena and are introduced with reference to real objects. Activities such as folding and cutting paper and drawing on a drawing sheet play an important role. In this way, geometric figures appear as real objects and geometric sentences formulate statements about these objects. The reason for their validity is usually the experience gained when operating with the objects (e.g. folding paper).

Terms are defined operationally, i.e. by specifying manufacturing instructions, for example:

- Folding lines are *straight lines.* Straight lines can be drawn with a ruler.
- If you fold a sheet of paper twice so that the parts of the first fold line coincide with each other on the second fold, the resulting lines are called *perpendicular to each other*.

The theorems of empirical geometry are deduced from activities or read off as the results of activities, e.g.:

- By folding, you can make a quadrilateral in which two neighboring sides are perpendicular to each other. Such a quadrilateral is called a rectangle. In a rectangle, opposite sides are the same length and parallel.
- A plane can be paved with triangles of the same shape and size. The angle sum theorem for triangles can be observed on such paving.

From grade 7 onwards, mathematical activity becomes more systematic, and conceptual thinking is required more than before. This also means that the conceptual ideas of geometric objects are made more precise in the sense of increasing idealization, and the first proofs of geometric theorems are carried out. A transition to a "Platonic view of geometry as a theoretical extension of human perception" (Tall, 2013, p. 61) is strived for and often implicitly implemented.

Müller-Hill (2015) has reconstructed the problem from a semiotic perspective. According to her interpretation, there are two possible conceptions of geometric drawing representations.

- *Image*: A sensually accessible object with properties such as color and size. This can itself be an "object" of empirical geometry.
- *Diagram* in the sense of Peirce: This stands for a "general object" or a class of objects that is determined by the structural relations between the individual components. It thus harbors a wealth of logical possibilities that must be suitably brought out in a proof.

An exact proof of the theorem that the perpendiculars of a triangle intersect at a point uses only those structural relations that are independent of the randomly chosen shape of the drawn triangle. During the transition from prescientific to advanced geometry, the semiotic status of graphic representations changes from an iconic image to a regularly used diagram.

This change, which was implicitly assumed by the trainee in the episode above, is often not realized by the learners. The learners initially use the representations almost exclusively in a pictorial way, in relation to the empirical reference range of drawing and folding sheet figures. In this context, it seems obvious that the perpendicular bisectors of a triangle intersect in a point, because all carefully made drawings indicate this. In advanced geometry, however, the intersection theorems for triangular transversals deserve argumentative justifications to ensure general validity. The illustrative use of graphic representations is no longer sufficient, a diagrammatic use of drawings is required instead. Müller-Hill (ibid.) points out that an appropriate use of dynamic geometry software could support the transition.

The transition from empirical geometry to Euclidean geometry contains an epistemological obstacle according to the notion developed by Bachelard (1938) and transposed into mathematics by Brousseau (1983). These obstacles are intrinsic difficulties of knowledge.

> The concept of epistemological obstacle gives Brousseau a way to interpret some of the recurrent and non-aleatorical mistakes that students make when they learn a specific topic. He claims that there is a logic behind these students' mistakes and explains them in terms of a knowledge that suffices to solve problems fruitfully but fails to appropriately solve others (Radford, 1997, p. 29).

The logic behind the learners' reactions is that of experience-based knowledge building, which doubly fails to recognize the significance of the evidence provided by the teacher:

1. Because of the experience gained through drawing, further proof is considered superfluous (“That they intersect at a point, you can see this.”).
2. The general validity of the argumentation is not recognized (“We see it for *this* triangle.”).

These reactions show that at least some of the learners did not follow the implicit transition to a theoretical level with its own questions and forms of knowledge formation. The informal objection “*What is interesting is whether they also intersect in the middle of the triangle*” could be the starting point for a fruitful discussion in which the following questions are examined: Why do the perpendicular bisectors of a triangle always intersect at a point, whereas the position of this point depends on the shape of the triangle? What is the “middle” of a triangle? How could this term be specified mathematically?

This transition from practical geometry to theoretical geometry in the Euclidean style is perhaps the biggest hurdle in secondary school geometry lessons. Analytical geometry requires a further striking step with the elaborate concept of the geometric vector, which is known to entail several conceptual difficulties. For its development, Tall (2013, pp. 60) assumes a sequence of advanced mathematical thought processes that are characterized by a general growth of sophistication.

- A vector as a translation in space in a certain direction by a given distance requires “a shift in thinking from a process occurring in time to a thinkable concept independent of time” (ibid., p. 62).
- Such a translation induces a whole collection of equivalent arrows with the same length and direction, and it can be represented by anyone of them. This requires the intermediate concept of equivalence class and representatives.
- A change in focus finally allows to imagine “a free vector that can be moved to show the translation of any particular point. [...] This single free vector is an embodiment that represents the translation as a whole.” (ibid., p. 69).

The further development of a formal geometry based only on set-theoretic definition and formal proof is usually reserved for the university and can become a problem of the first discontinuity.

The sequence of ‘practical geometry’, ‘theoretical geometry in the Euclidean style’ and ‘analytical geometry’ mark three essential stages within which finer graduations can exist. Van Hiele (1986) has shown this impressively for the stage of practical geometry, and his findings have often been taken up. He observed, for example, that a figure initially is seen as a whole (e. g. an equilateral triangle has equal sides *and* equal angels). The insight that one property of a figure can lead to others develops gradually. So, in the context of ruler and compass construction a surprising discovery can be made: Only equal sides of a triangle are involved in the construction, but you see equal angels in the result.

Within the stage of 'theoretical geometry' the concepts of area and similarity are subject to various steps of sophistication.

Bridging the second discontinuity in the domain of geometry includes the requirement to become awsare of refinements and transitions between different stages of knowledge and inherent obstacles and the necessity to develop means to make the progress explicit in the classroom.

3.2 Stages of knowledge formation in arithmetic and algebra

In my first attempt at teaching negative numbers, I worked with a textbook that explained addition and subtraction using vectors. Numbers were represented by arrows determined by length and direction. Calculations were justified by vector operations with certain "foot-and-tip-rules" which resulted in pictures like figure 4.

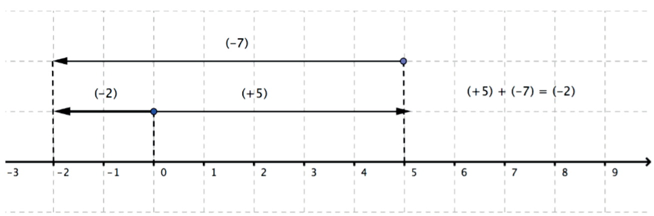


Figure 4. – Arrows representing addition of whole numbers.

I had intuitive concerns, but couldn't explain why, and decided to try the approach. Addition still worked, but when subtraction produced an image like figure 5, a student bright enough to handle the rules correctly came forward and said from the deepest conviction: "There must be another way to explain this."

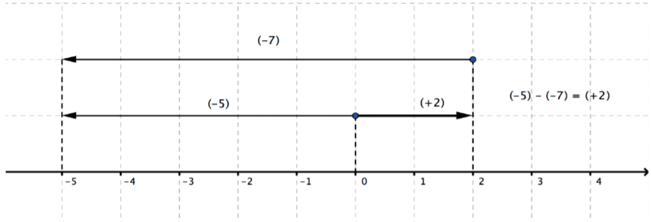


Figure 5. – Arrows representing subtraction of whole numbers.

> *I went home very thoughtful. If you look at the left side of the equation, you immediately notice that "you can't get more minus." Why on earth should a positive result then come out on the right and why should one believe the arrows of all things?*
>
> *Later, I treated this approach in a seminar. The student who had to present it explained the calculations and then said quite honestly: I know how to calculate, and then I set the arrows accordingly. At this point, the relationship between illustration and explanation is twisted: it is not the illustration that helps to explain a fact, but one uses what is to be explained to understand the illustration supposed to explain it. The student brought to light that there was obviously an "anti-didactic inversion" in the sense of Freudenthal (1963, p. 19).*

While negative numbers are unproblematic from a professional point of view, the difficulties of teaching them can easily be underestimated. The advanced development of a formal understanding of numbers and number operations is essential for the development of negative numbers. Freudenthal (1989) emphasizes that the intellectual origin of negative numbers was not the description of reality, but the formal-algebraic need to have uniform solution schemes for equations. However, the interpretation of rational numbers as relative numbers with respect to a reference mark, which corresponds to zero on the number line (Winter, 2016), was developed late in history by Gauss. It attempts to unite the formal-algebraic conceptual core and a descriptive use and to consider extended possibilities of measurement through directed quantities.

However, the conflict between negative numbers and an elementary notion of magnitude accompanied the history of negative numbers for centuries. The idea that "nothing" cannot be diminished by subtraction leads to the natural assumption that it makes sense to extend the subtraction of positive integers by putting $m - n = 0$ *for* $m < n$. This idea was related to the view of zero as absolute zero with nothing "below" it and hindered the conception of a uniform number line (Hefendehl-Hebeker, 1991).

Interestingly, Radford (1997) points out that these difficulties are partly culturally determined and were particularly typical of Western medieval mathematicians, whereas Chinese mathematicians knew how to overcome them early through a cleverly chosen system of representation consisting of red and black counting rods for positive respectively negative numbers. He suspects that a difference in basic philosophical attitudes is the reason, because the principle of duality (yin-yang) plays an important role in Chinese philosophy and "opposition and equivalence" belongs to "the real core of the Chinese episteme" (ibid., p. 31). So, Radford concludes (ibid., p. 29): "Thus, the difficulty that positive numbers pose to the rise of negative numbers is not an intrinsic problem of knowledge. It depends upon the local, cultural ideas about science, mathematics, their objects and methods."

Despite these relativizations, negative numbers contain various epistemic difficulties and remain a challenge for learners at lower secondary level. The transition from thinking in absolute values to relative numbers requires concept changes, just like the transition from natural numbers to fractions. The minus sign suddenly appears in three different meanings (sign of a number, operation sign, inversion sign), and the relationship between these meanings can be confusing. Addition no longer automatically leads to an increase, and subtraction does not necessarily lead to a decrease. The multiplication of two negative numbers can no longer be explained with object-related ideas.

Moreover, presentation contexts often are not self-explanatory. Dealing with illustrations always moves in a field of tension between reading off and reading into, and the demands of reading into should not become too strong. Thus, the use of the vector model in the episode above contained a double difficulty for the learners:

1. It was difficult for them to handle.
2. It also did not provide a credible guarantee that the calculations were reasonable.

The vector model certainly has its purpose, but perhaps not as an introduction to the subject. In the context of vector algebra, it might be used to demonstrate what Freudenthal (1989) emphasized and where he sees a decisive benefit of the existing calculation rules: “Algebra fits geometry.”

As in geometry, the teaching of arithmetic and algebra is also subject to a development from empirical to theory-based thinking. Growing demands on broadening one number system to a larger one with richer properties and inherent epistemological obstacles form a web of difficulties. Operations experienced in arithmetic are finally generalized in symbolic algebra. This achieves a new level of abstraction.

In elementary school and at the beginning of lower secondary school, natural numbers and the associated number operations are developed from action knowledge and supported through imagined actions. Numbers have to do with counting and quantifying. Addition is represented as counting on, adding, increasing; subtraction appears accordingly as counting back, taking away, decreasing, and is finally also interpreted in terms of complements and differences. Multiplication is taking multiple times, more rarely also combining, division is dividing and measuring. Of course, these activities and especially the algorithmic abilities to be learned require their own efforts and they are also susceptible to misconceptions. However, a content-related understanding of the basic arithmetic operations is formed. Learners can usually imagine a realistic situation for a number operation. However, the aim here is to initiate an awareness of numbers as ideal objects that obey strict arithmetical laws.

A new experience in dealing with numbers is introduced through the treatment of fractions. When introducing fractions, there is still a range of content-related interpretations that are based on the real world and can be supported by everyday knowledge. Nevertheless, the concept of fractions is a difficult one with many different manifestations (see 2.1) and two different symbolic representations (fractional and decimal notation). The requirements to recognize and use the structural similarities are correspondingly high. Finally, the ideas of arithmetic operations as well as connotations of their accompanying effects must be appropriately expanded or even modified at this new level in the structure of the number system. Careful didactic efforts are needed to ensure that learners do not lose contact with the meaning and significance of what they are doing (see 2.2).

The last step is to strip away the concrete references to magnitudes and the measurement of magnitudes and to consider fractions as pure "rational numbers".

Negative numbers are primarily motivated by formal algebraic needs. The challenge for the teacher here is to make this necessity and the associated reasoning understandable, but also to give an outlook on the mathematical benefits of this approach.

The treatment of irrational numbers and the phenomenon of incommensurability once again requires a new level of approach to mathematics. For learners, the relationship between mathematics and reality at this point finally becomes different from what they are used to. The introduction of irrational numbers cannot be justified by practical issues, it is motivated by the pursuit of precision and theoretical coherence.

> The introduction of real numbers cannot be justified by practical measurement tasks. In real situations, especially in measurements, irrational numbers never occur directly. The decision as to whether a measured number or a solution to an equation is rational or not cannot be made experimentally or empirically, not even by computer calculation, but only by theoretical argumentation. The transition from rational to real numbers is an extension of the number range that is expedient for theoretical reasons. It ensures that for certain geometric and algebraic problems … clearly existing solutions also exist in theory as well-determined objects (Kirsch, 1994, S. 90; translated by the author).

Researchers agree that the historical trigger was a new relationship to mathematics. It manifested itself in reflection on the knowledge gained from practice and in the need to explain and justify, which we regard as the origin of scientific thinking with far-reaching consequences (Mittelstraß, 2014, p. 275). In his Metaphysics, Aristotle (384 - 322 BC) praises incommensurability as the prototype of a scientific discovery gained from the pure pursuit of knowledge and shows that people can expand their knowledge solely through deductive thinking (Artmann, 1999, p. 229 f.). The possibilities of thinking itself were therefore a key factor in this progress.

> One of the greatest powers of scientific thinking is the ability to uncover truths that are visible only 'to the eyes of the mind', as Plato says, and to develop ways and means to handle them. This is what Euclid does in the case of the irrational, or incommensurable, magnitudes… (Artmann, 1999, Preface).

Despite all the achievements of antiquity, which led to the discovery and study of incommensurable quantities, centuries had to pass before the irrational was recognized as a number. The irrational was not only the negation of the rational, but it was also the "inexpressible, incomprehensible, imageless" (Sonar 2016, p. 47) or indeterminable, as expressed in the Greek adjective *arrhaetos* and its Latin counterpart *surdus.*

All in all, there is something mysterious about irrational numbers. One can logically deduce that the system of rational numbers is insufficient, for example, to solve rational equations or to assign a number to every point on the number line. Irrational numbers can be represented symbolically by letters such as e and π or by operation signs such as $\sqrt{2}$. However, they cannot be represented exactly in the decimal place value system, but only with a desired precision. The rest disappears in the sense of M. Stifel "under a fog of infinity" (Stifel, 1544, p. 103). Thus, the treatment of irrational numbers is also associated with fundamental mathematical ideas such as infinity and approximation.

In this summary, it becomes clear how much a treatment of irrational numbers in the classroom can enrich the image of mathematics. The richness lies in the range between mathematical-logical inevitabilities and deep philosophical questions such as the problem of the continuum or the relationship of mathematics and its concepts to reality (Bedürftig and Murawski, 2015, p. 4). But to reveal this richness in the classroom, teachers need to develop an awareness of its range and its intrinsic tension.

For the development of the number system in mathematics lessons, teachers need an awareness of the progressive refinement of the subject and its epistemological and philosophical aspects.

A particular didactic challenge is the initiation of algebraic thinking and practicing the elementary algebraic formula language. This symbolic language, which emerged in 16th century in Europe and became a constitutive tool of mathematics, is used as a matter of course in university mathematics. As such, it is beyond question, even though the requirements for skillful handling and a pronounced "symbol sense" (Arcavi, 1994) can also become a problem here.

Algebra therefore is a particular challenge for teachers and learners.

> "Algebra involves a new way of thinking that requires substantial change of perspective, alongside the considerable manipulative rule-based skills required to deal fluently and accurately with its notation. Students come to algebra after learning arithmetic, which of course provides essential skills for doing algebra. Perhaps surprisingly, however, the transition from arithmetic to algebra is not a smooth one." (Arcavi et al., 2017, p. 72).

According to Radford (2010), two aspects form the core of the above mentioned "new way of thinking":

- Algebra deals with objects of an indeterminate nature.
- This is done in an analytical way, i.e. one operates with indeterminate quantities that cannot be processed numerically, but are thought of relationally, i.e. in their relations to each other and to known quantities.

To a certain extent, algebraic thinking can already be initiated in elementary school arithmetic and in the early stages of secondary school. Wittmann (2021) describes learning environments in which children discover arithmetic patterns using concrete number examples and describe them pre-algebraically, i.e. without using symbolic language. This activity requires a new view of arithmetic expressions that is not only focused on calculation results but also on structural aspects.

In the best-case scenario, operational abstractions (Tall, 2013) are already achieved in this context. This means that learners can consider an arithmetic expression such as '3+4' in two different ways: as an instruction to calculate a result or as a designation for the result of the calculation, here a sum with a certain construction plan. In the first case, the focus is on a process (addition), in the second on a mental concept (sum). This is a new cognitive stage for which Tall (2013) coined the term 'procept'.

At the beginning of a systematic treatment of algebra in secondary school, these approaches are often taken up. The formula language is introduced to describe non-mathematical and internal mathematical patterns, e.g. sequences of numbers or geometric figures, and their inherent arithmetical laws of formation. At this stage, the algebraic terms are used as shorthand descriptors and can be used to calculate further sequence elements.

The decisive factor here is that variables as indeterminates are given a concrete meaning in the respective context and that the syntax of the descriptive terms can be interpreted as a certain way of looking at the respective situation. Term equivalence is articulated as equality of results (different terms each lead to the same calculation result when inserting numbers) or equivalence of description (different terms represent correct descriptions for the structure of the figures). However, the transition from these motivating beginnings to purely formal calculation is not a self-starter and is easily underestimated in terms of the complexity of the requirements.

Observations have shown that many learners were not yet able to develop sufficiently general concepts of terms and variables based on the introductory situations. The expressions remained attached to the original situations experienced, i.e. they were still considered as descriptions of these situations and not as elements of a symbolic area of its own right which can be manipulated according to fixed rules (Radford 2010). In addition, many learners were not sufficiently aware of the basic laws of arithmetic that they could act as a guide to the algebraic calculus. Explicit mediation steps are therefore necessary in the learning process on the way from concrete introductory situations to general concepts. (Hefendehl-Hebeker and Rezat, 2023, p. 149). The detachment from concrete references again marks a new stage.

Finally, learners must learn to understand and handle the algebraic symbolic language as a world of its own and develop *structure sense* (Linchevski and Livneh, 1999) in this process. To be able to operate flexibly and effectively at the symbolic level, a pronounced sense of term structures is required, which is oriented towards the arithmetic architecture and not purely on surface features. Essential is the ability to bundle subterms in an algebraic expression into units in such a way that a usable basic structure becomes recognizable. For example, the term $u^2v^4 + 2uv^2w + w^2$ can be transformed into $(uv^2 + w)^2$ according to the 1st binomial formula if the product uv^2 is seen as a unit. Structure sense is a crucial stage in the development of algebraic ability.

The highest stage is reached when algebra can be used meaningfully as a tool. Arcavi (1994) coined the term *symbol sense* for this ability. This includes a flexible skill in transforming terms, including the abilities previously described under the term structure sense, and further abilities, for example an aesthetic sense of the power of formula language.

Teachers in algebra lessons must therefore find their way back from a professional approach to the sensitivities of beginners, who can only gradually approach this goal in small steps, and they must become aware of important requirements and stages along the way.

4. Driving questions of mathematical knowledge building in elementary context

Some years ago, I gave my student teachers the following task (inspired by Affolter et al., 2002, p. 28)[3]*:*

> Imagine you have 64 cubes with an edge length of 2 cm. Build cuboids from them so that all the cubes are used up.
> a) Determine the volumes of these cuboids.
> b) Which of these cuboids has the smallest surface area? Determine it in cm^2.
> c) Which one has the largest surface area? Determine it in cm^2.
> Solve the task at the presentation level of a teacher who wants to prepare for dealing with the task in class.

Most students did not go beyond listing the possible cases according to a more or less clever system and calculating the corresponding surfaces. They did not formulate more than the immediate answers to the task questions and did not differentiate between a desirable student solution and preparatory considerations by the teacher.

3. The discussion of this example follows Hefendehl-Hebeker (2015).

However, if you want to fully utilize the task's substance in the classroom, a clever list of possible cases, as shown in Table 1, is helpful.

Table 1. – Possible cuboids constructed of 64 cubes

Edge lengths, measured in cube edges	**Surface area, measured in cube surfaces**	**Surface area, measured in cm²**
(1, 1, 64)	$2 \cdot (1 + 2 \cdot 64) = 258$	1032 cm²
(1, 2, 32)	$2 \cdot (2 + 32 + 64) = 196$	784 cm²
(1, 4, 16)	$2 \cdot (4 + 16 + 64) = 168$	672 cm²
(1, 8, 8)	$2 \cdot (8 + 8 + 64) = 160$	640 cm²
(2, 2, 16)	$2 \cdot (4 + 32 + 32) = 136$	544 cm²
(2, 4, 8)	$2 \cdot (8 +16 + 32) = 112$	448 cm²
(4, 4, 4)	$2 \cdot (16 + 16 + 16) = 96$	384 cm²

The following aspects can be considered:

- The possible triples of side lengths are ranked lexicographically. This arrangement is both a means of representation and evidence. If carried out correctly, it guarantees that all cases have been recorded.
- The entries in the middle column can be understood as a calculation of the surfaces for the case that the building blocks are unit cubes. The results can then be easily converted for each cube size.

Further observations can be made in this table:

- The cube has the smallest surface area.
- The cube snake, consisting of 64 cubes, has the largest surface area.
- The arrangement of the dimensional numbers for the surfaces runs in the opposite direction to the lexicographical arrangement of the number triples for the side lengths.

The observed regularities raise the question of "Why?". Reasons can be discovered through action by looking for methods to transform the individual cuboid shapes into one another. The (1, 1, 64)—snake is transformed into the (1, 2, 32)—plate by halving the snake and placing the halves next to each other. This conceals cube faces that were previously visible. The surface area thus becomes smaller because of this reconstruction. You can continue in this way step by step until you have reached the cube shape.

Such preparation leads directly to impulse questions for the lesson, which can be asked once the pupils have dealt with the task and have collected their solutions:

- Try to sort the possibilities you have found.
- What criteria did you use to sort them?
- How can we be sure that all possibilities have been found?
- Compare the data sets for the side lengths and the surfaces. What stands out?
- How can we convert the (1, 1, 64)—cube snake into a (1, 2, 32)—square as easily as possible? How can you see that the surface area becomes smaller during this conversion?
- What happens if you carry out the experiment with cubes of a different size?

With such questions, more mathematical substance can be brought to light than a simple solution to the problem reveals. They are oriented towards authentic processes of mathematical knowledge building, the motives that drive them, and the supporting thought processes: experimenting, observing, representing, interpreting, systematizing, securing, justifying, and generalizing …. Within this framework, cognitive activation of learners at different levels with meaningful intermediate conclusions is possible. Here, something of what Toeplitz (1928, p. 6) called the "gears of a mathematical theory" emerges.

5. Conclusion

These considerations should show that a young teacher must gain a new sensitivity for the development of mathematical knowledge beyond the higher standpoint acquired at university. He or she must develop an epistemological awareness (Prediger and Hefendehl-Hebeker, 2016) of the stages of knowledge formation, their forms of articulation and accompanying difficulties. Such a disposition could be awakened and promoted through epistemologically oriented courses in which vignettes of the kind presented in this article play a prominent role together with the literature used for their interpretation.

However, this awareness is not only an important aid to practical teaching, but it can also help to see different stages of mathematical thinking from the first experiences up to the "higher standpoint" under a unifying developmental aspect and thus to form a conception of the "unity of the two mathematics" at school and university (Toeplitz 1932, p. 1) within which each stage of development has its own dignity. Such a disposition could be initiated and promoted by subject-specific longitudinal sections that are supplemented by epistemological aspects. These longitudinal sections could relate to individual topics, e.g. the isoperimetric problem (Bauer, 2013), or develop comprehensive mathematical ideas, e.g. the extensions of the number system or measurement in geometry (lengths, surface areas, volumes). In this way, it could become clear how the creation of mathematical tools of increasing scope and a continuous sophistication of mathematical thinking correspond to each other.

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