From global to local aspects of Klein's second discontinuity

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The global question of how to identify, develop and assess mathematical knowledge that is relevant to future secondary school teachers, has been central in the emergence of mathematics education research from early on. We review parts of this history from the viewpoint of the anthropological theory of the didactic, and in particular the notion of relationships to mathematical praxeologies that are held by certain positions within school and university institutions. We also consider a modern case, where the questions arise in a very practical sense: how to bridge the gap between standard undergraduate mathematics courses and a school relevant model of real numbers and functions? We show how both theoretical and practical aspects of this more local question arise in a so-called capstone course for students with about two years of undergraduate mathematics experience.

Keywords: mathematics teacher knowledge, infinite decimal representations of real numbers, capstone courses, Klein's second discontinuity

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De aspectos globales a locales de la segunda discontinuidad de Klein

La cuestión global de cómo identificar, desarrollar y evaluar el conocimiento matemático que es relevante para futuros profesores de secundaria ha sido central en la emergencia de la investigación en educación matemática desde sus inicios. Revisamos partes de esta historia desde el punto de vista de la teoría antropológica de lo didáctico, y en particular la noción de relaciones con las praxeologías matemáticas que son sostenidas por ciertas posiciones dentro de las instituciones escolares y universitarias. También consideramos un caso moderno, donde las preguntas surgen de manera muy práctica: ¿cómo cerrar la brecha entre los cursos de matemáticas de pregrado estándar y un modelo de números reales y funciones relevante para la escuela? Mostramos cómo tanto los aspectos teóricos como prácticos de esta cuestión más local surgen en un curso de culminación para estudiantes con aproximadamente dos años de experiencia en matemáticas de pregrado.

Palabras-claves: conocimiento del profesor de matemáticas, representaciones decimales infinitas de números reales, cursos de culminación, segunda discontinuidad de Klein

La double discontinuité de Klein : de perspectives globales à des perspectives locales

La question globale d'identifier, développer et évaluer les connaissances mathématiques qui sont pertinentes pour les futurs enseignants du secondaire, a été depuis les débuts un levier central dans l'émergence de recherches en didactique des mathématiques. Nous exposons des éléments historiques de cette question du point de vue de la théorie anthropologique du didactique, et en particulier la notion de rapport aux praxéologies mathématiques entretenu par certaines positions au sein des institutions scolaires et universitaires. Nous examinons aussi un cas moderne où ces questions apparaissent d'une manière plus pratique : comment combler le fossé entre une licence générale en mathématiques et des conceptions des nombres réels et des fonctions d'une variable réelle qui sera pertinente pour l'enseignement secondaire ? Nous montrons comment les aspects théoriques et pratiques de cette question plus locale apparaissent dans un cours de synthèse pour des futurs enseignants, qui ont passé deux ans de cours mathématiques universitaires.

Mots-clés : connaissances mathématiques d'enseignant, représentation décimale de nombres réels, cours de synthèse pour enseignants, seconde discontinuité de Klein

Introduction

We consider that the following question indicates a central *raison d'être* of the Didactics of Mathematics (across all of its variations):

> Q_0: *What knowledge must mathematics teachers have in order to deliver good teaching?*

The question is evidently broad and imprecise, most notably due to the undefined meaning of "good". It is also clear that more precision is needed to obtain a question that could have scientific answers. But both teachers, researchers and even broader groups would recognize some meaning in Q_0. They might also agree that the links between research in Didactics of Mathematics on the one hand, and mathematics teacher education on the other, are both strong and old, and come from the expectation that the former could produce knowledge that is useful to the latter and hence, at least in some sense, the kind of knowledge that Q_0 asks about.

Of course, teacher knowledge is in general a very complex object. Few would deny that it involves *professional* components that need to be acquired through practice. On the other hand, few societies assume today that teacher knowledge can be acquired exclusively through practice; in other words, they establish some form of "initial" education. For the teaching of academic subjects like mathematics, this initial education almost invariably involves this subject matter in some form. And it is generally considered a truism that teachers should possess a solid knowledge of the subject they teach, *in casu* mathematics.

As we shall see later, it could be said that the Didactics of Mathematics was born from the realisation (or at least the conviction) that "mathematical knowledge" is part of the answer to Q_0, but that the following subquestions are non-trivial:

> Q_1: *What mathematical knowledge is necessary (or just relevant) for mathematics teachers to deliver "good" teaching? How is it best acquired and certified?*
>
> Q_2: *What other forms of knowledge (if any) are necessary? How are they best acquired and certified?*

Again, these questions clearly lack precision, but we can now formulate the overall aims and structure of this paper:

- first, provide a theoretical framework for the study (including more precise formulation) of Q_1, based on the Anthropological theory of the Didactic (ATD),

and then use this framework to:

- outline main trends of existing methods and answers for Q_1 that can be found in the international research literature,

- analyse more deeply the problem of task design for pre-service teacher education (as a partial way to answer Q_1) illustrated by some cases of tasks developed at the University of Copenhagen, in relation to prospective secondary level mathematics teachers' knowledge about real numbers.

The last point constitutes the main part of the present paper, which is mainly theoretical (with the case illustrating and generating theoretical points). At the end we return to the meaning of this particular problem and case within the broader context of Q_1.

Note that in this paper, we consider only the (needs for developing) *mathematical* knowledge of *prospective* teachers. This is in no way to be construed as a denial of the relevance of other forms of knowledge or of the professional knowledge developed in and through teaching practice. We also recognise that it is not possible or productive to fully isolate or delimit "mathematical" components of mathematics teachers' knowledge within their theoretical and practical knowledge at large. Nevertheless, there are important and researchable problems related to Q_1 which are specific to the selection, delivery and assessment of mathematical knowledge within (initial) mathematics teacher education—and it is on some aspects of these that we focus here.

Framework and research questions

Any society that certifies individuals for teaching mathematics at a given level will furnish practical answers to Q_0, Q_1 and Q_2, at least (but not always limited to) the knowledge required at the entrance of the profession. We can consider these answers as collections of *relationships to* (knowledge) *objects* O_k to which the mathematics teacher t within a certain school institution S must hold a certain relationship $R_S\left(t,O_k\right)$ to, i.e. some collection of type

(1) $$\bigcup_{k\in K_S} R_S\left(t,O_k\right)$$

(cf. Chevallard, 1992), where K_S is a finite index set. This collection may in principle be empty, if no requirements are present; but even if no initial teacher education exists, other requirements (such as t having previously occupied the position P as pupil in some school institution S', where $S'=S$ is possible) with more or less specified relationships $R_{S'}\left(p,O_k\right)$ obtained to some objects O_k, could still be stipulated. Even in this case, a special institution I — which is typically, but not always, a teacher education institution of some sort — may be endowed with the power to decide whether or not an individual y has the relationships required to occupy the position t in S. In principle, the question is then whether the relationships of y to O_k are sufficiently near $R_S\left(t,O_k\right)$ for all $k\in K_S$.

However, in practice, due to a great distance between I and S, it is common that I replaces $\{R_S(t,O_k) \mid k \in K_S\}$ by $\{R_I(y,\omega_k) \mid k \in K_I\}$, where $\{\omega_k \mid k \in K_I\}$ consists of objects used by I to certify officially that y satisfy the said requirements (and we use the letter ω, instead of O, to stress the change of institution). And then society will assume that if

$$\bigcup_{k \in K_I} R_I(y,\omega_k) \tag{2}$$

is affirmed by I for some individual y, then that person can be admitted to position t within S, and we may assume, or merely declare, that (1) is then satisfied. Concretely, (2) is often determined by y passing a certain number of tests within I, each determining whether a certain number of relationships $R_I(y,\omega_k)$ are satisfactory. This is to some extent the case also for the tasks for future teachers, presented later in this paper, even if they are designed deliberately to relate to some O_k. Anyone with any experience of current teacher education (or certification) systems will know that while (2) is more or less concretely specified by the regulations within I, the relation of (2) to (1), and also (1) itself, are often far from transparent. Moreover, (1) develops throughout the career of a teacher in position t, and this may well lead to initial inadequacies being remedied.

Nevertheless, we cannot assume or claim from the outset that (2) is completely arbitrary with respect to (1). In particular, when it comes to *mathematical* objects ω_k met by y within I, some are indeed likely to be closely related to mathematical objects O_k met by pupils p and teachers t within S. To identify and question such cases, at least locally, is the main idea of this paper when it comes to addressing Q_1 in practice, following up on our previous work (Winsløw and Grønbæk, 2014).

To do so, we need a less abstract way to describe the "objects" of type O_k and ω_k. In ATD, knowledge objects—in particular, elements of mathematical knowledge—are more recently modelled, within ATD, as *praxeologies*, consisting of praxis and logos blocks (Chevallard, 1999). We will also consider, from this point on, the frequent case (cf. below) where the institution I educating and certifying teachers is a university institution U.

Considering now the special case of mathematical praxeologies $\{\omega_k \mid k \in K_U\}$ for which U requires future teachers to hold relations $R_U(\sigma,\omega_k)$ in view of their pertinence to some school mathematical praxeology O_k, we may consider the passage (or rather, possible relations) of type

$$R_U(\sigma,\omega_k) \to R_S(t,O_k) \tag{3}$$

where the arrow merely indicates the chronological order in which an individual may occupy the positions σ and t as student within U and teacher within S. The mathematical praxeologies, to which the individual relates in these positions, are in principle different.

Even the mathematical praxis and logos required from the teacher in relation to some O_k, in which her pupils are to engage, may be quite different from the relation aimed at for the pupil, as when the teacher is supposed to pose and correct exercises for the pupils. In other words, (3) can be used on very specific cases of the relation between (1) to (2), typically singled out because $R_S(t, O_k)$ is of some importance, and can be expected to be related to $R_U(\sigma, \omega_k)$, due to O_k and ω_k being somehow related mathematical praxeologies. That such impact and relatedness may be relatively absent—not only locally, but in a more general sense—is what Klein (1908) singled out as the *second discontinuity* afflicting modern organizations of mathematics teacher education (cf. Grønbæk and Winsløw, 2014): both the university student and the active teacher may perceive little or no impact or relatedness of the kind just defined. On this theoretical background, we can now develop the initial question Q_1 into the following research questions which, although they are likely far from covering all aspects one could see in Q_1, are at least amenable to research, for fixed institutions *S* and *U*:

> **RQ1**. Given a central praxeology *O* to be taught in *S*, how can some $R_U(\sigma, \omega)$ be used to build school relevant $R_U(\sigma, O)$?
>
> **RQ2**. What needs exist to develop $R_U(\sigma, \omega)$ further (into what we shall later call $R_U^*(\sigma, \omega)$), in view of contributing to $R_U(\sigma, O)$? These questions do not adopt the global viewpoint indicated by (1) and (2), as they focus on "central" instances of praxeologies. This implies a methodology of case studies (constrained by institutions and specific choices of the praxeological instances). However, some of the previous international research, which we review in the next (background) section, has in fact adopted the more global viewpoint. We discuss how these studies contribute to answer the questions above, or at least to motivate them.

We emphasize that this paper is essentially a theoretical paper, where cases are used to generate hypotheses and illustrate the more general research questions outlined above; by cases we mean instances of mathematical praxeologies, and to some extent concrete institutional contexts.

Synthesis of global positions and results

Klein's heritage

As exposed in some detail by Winsløw and Grønbæk (2014), Felix Klein was one of the first who problematized the passage from university studies of mathematics to teaching at what we would now call secondary level, beginning with his inaugural address as professor in Erlangen in 1872. The life, work and legacy of Klein—particularly within mathematics education—has been reviewed in larger depth within a recent book edited by Weigand, et al. (2019). Kilpatrick (2019, p. 215) notes, in his chapter within this book, that

> Klein's courses for teachers were part of his efforts to improve secondary mathematics by improving teacher preparation. Despite the many setbacks he encountered, no mathematician has had a more profound influence on mathematics education as a field of scholarship and practice.

We note here the strong link between teacher education and the birth of "mathematics education as a field of scholarship", also stressed in our introduction of Q_0 at the outset of the present paper. It was Klein's personal and institutional efforts to improve the preparation of secondary mathematics teachers that first led him to reflect, more broadly, on the needs and nature of mathematics education. His influence in this regard stretches far beyond his own environment and time, most famously through the foundation of the International Commission on Mathematical Instruction, for which he served as the first President from 1908 to 1920.

His ideas on mathematics teacher education have also exercised a more practical and longstanding influence, not least through the use of his lecture course for future teachers both in Germany and in other countries (for a recent translation into English, see Klein, 2016). The main idea of these lecture notes was interpreted, by Grønbæk and Winsløw (2014), in terms of (3): to develop prospective teachers' relationship $R_U(\sigma,\omega)$ with "higher mathematics" (Klein's term) through specific courses at the university, in view of becoming more useful for them as teachers, in other words, to enrich relationships of type $R_S(t,O)$. In Klein's work, Q_0 and Q_1 are not sharply distinguished, and while the possibility of ruptures between (1) and (2) is broadly recognized, Klein clearly saw it as a task for universities to bridge it through adaptations of (1), not so much to some inert version of (2) as to the service of a secondary school mathematics curriculum which would also have be updated in the light of recent developments of "higher mathematics". Klein clearly saw that such an endeavour would require a strong commitment of university mathematics teachers not only in teacher education, but also in contributing more directly to the development of secondary school mathematics; his own efforts in this direction were many-sided and influential as well, as documented by several chapters in (Weigand et al., 2019).

Now, a century later, we can notice both successes and apparent failures of this programme. "Higher mathematics", in the sense of courses whose content is roughly selected from what form the bases of current scholarship in pure mathematics—continues to be a main ingredient in secondary mathematics teacher education in many countries. Sweeping reforms of mathematics education curricula, both at university and in schools, were carried out in the 1960's and 1970's, under the label "New Math". The outcomes continue to be analysed and debated (see, for instance, Kline, 1973, for an early, and naturally controverted, contribution). While it is impossible to know what Klein's view on these later reforms would have been, it is certain that the fundamental distance between the mathematical sciences (not limited, by the way, to pure mathematics) and school mathematics has not ceased to grow. University mathematics curricula have remained surprisingly stable since the 1960's—notwithstanding later adaptations, most notably to include newer developments in statistics, computing and discrete mathematics (cf. Bosch et al., 2021). At the same time, reforms of school mathematics have been frequent, deep and strongly debated in many countries of the world, both before and after the period of New Math, and in many different directions.

Despite the necessary brevity of this outline, there is no doubt that the problem for mathematics teacher education which Klein identified, remains of strong actuality. It is, roughly, the non-trivial character of Q_1 in institutional set-ups where university mathematics is strongly involved, as it continues to be in most Western countries (OECD, 2014). Some of Klein's concrete proposals are also relevant to answering RQ1-RQ2, as we shall touch upon later for the special case of the mathematics surrounding the concept of real numbers.

Qualitative and quantitative research on Q_1 and Q_2

The more global questions introduced above have been the subject of both theoretical, qualitative and quantitative research, at least since the late 1960's. A famous early contribution was Begle's (1972) study of how teachers' knowledge of abstract algebra correlated with the knowledge on school algebra of their 9th grade students. Begle (1972, p. 14) concluded that:

> …teacher understanding of modern algebra (groups, rings, and fields) has no significant correlation with student achievement in algebraic computation or in the understanding of ninth grade algebra. Teacher understanding of the algebra of the real number system has no significant correlation with student achievement in algebraic computation. However, teacher understanding of the algebra of real number system does have a significant positive correlation with student achievement in the understanding of ninth grade algebra. Nevertheless, while this correlation is statistically significant, it is so small as to be educationally insignificant.

These first results are still sometimes cited without the reservations and limitations that the author himself points out—such as the fact that the involved teachers were voluntary participants in a summer school on mathematics, and therefore not likely to be representative of 9th grade teachers at large. Nevertheless, these first results challenged the assumption that teachers' more extensive record of higher mathematics courses will automatically result in better teaching, reflected through the knowledge of their students in theoretically related fields of school mathematics. This largely confirms one of Klein's basic claims that the impact of academic courses cannot be taken for granted. Follow-up studies with somewhat less biased samples of teachers, such as Eisenberg's (1977), broadly confirmed this point, but also strengthened one of Begle's (1972) explicit hypotheses: that there might be "a lower bound of knowledge, below which the relationship between teacher knowledge and student performance does hold" (Eisenberg, 1977, p. 221).

This hypothesis, together with the possibility of other measures of "teacher knowledge of mathematics" correlating with student knowledge, was since examined further. An interesting study of the cited hypothesis—with a much more global scope than the case of abstract algebra and school algebra—was carried out by Monk (1994). He examined correlations between the number of academic mathematics courses taken by secondary level mathematics teachers, and their students' performance gains. Monk did in fact find a positive correlation with students having taken up to about 5 courses (a minimum largely exceeded by current undergraduate requirements in many countries). This suggests—with multiple caveats—that a minimal undergraduate mathematics background, formed by up to a year of full-time academic mathematics study, does have a positive effect on the teachers' efficiency, but that anything beyond that may have little or no effect. Naturally, as with all quantitative studies of correlations, many other variables could possibly have significant explanatory value, and at least to some extent put the suggested "positive effect" into question.

The question of *how* to define, and possibly measure, relevant forms of teacher knowledge, is latent in Q_0, Q_1 and Q_2, and more explicit (and limited to mathematical praxeologies) in RQ1-RQ2. Quantitative studies will eventually make choices along these lines, as when items are formulated for use in a test (where a relationship $R_I(x, O)$ of some member of I to some O is assessed based on how x solves one or more tasks pertaining to O). The question then arises, especially for studies of more global categories of knowledge: what relation exists between the inventory of tasks proposed, and a qualitative or theoretical definition of the categories?

Indeed, inventories of items have recently been constructed and used in major international studies of how student and mathematics teacher knowledge correlate, along with categories of knowledge (relevant to Q_1 and Q_2) that are defined in careful, yet quite general terms. A major centre for research in this area has been the University of Michigan, where an elaborate theorization of *mathematical knowledge for teaching* (MKT) was used, at the dawn of this millennium, in a large-scale investigation of primary school teachers' MKT and found strong correlation with their students' mathematical achievement, even when controlling for other plausible factors (Hill et al., 2005). Moving to international comparative studies, these ideas and methods were further refined and subsequently deployed in the "Teacher Education and Development Study in Mathematics" (TEDS-M) study, which involved 17 countries (Tatto, 2013). The results from this study are very rich and complex—including comparisons of teacher education programmes across and within countries—and cannot be subsumed in a few phrases. We shall however note two points, in the words of some of the main specialists:

> For secondary programs the most important influence on knowledge for teaching is the opportunity to learn university level mathematics (...) and the opportunity to read research in teaching and learning. (...) Teacher education programs' quality of opportunities to learn—as measured by their association with high levels of mathematics teaching knowledge, coherence on program philosophy and approaches, and internal and external quality assurance and accountability mechanisms, are all features that seem to contribute to increased levels of mathematics knowledge for teaching among future teachers. (Kraineret al., 2015, p. 118)

Closer studies of the most successful mathematics teacher education programmes for (lower) secondary school, carried out by Schmidt et al. (2013, p. 5), further identified course elements which these seem to share to a high degree; these include six standard undergraduate mathematics units (beginning calculus, calculus, multivariate calculus, differential equations, linear algebra, probability) along with three units on school mathematics education (math instruction, observing math teaching, functions). These programmes naturally all contain more elements; but this "core" is important to note. It is hard not to notice the consistence with Monk's early results and also with Klein's contention that school-oriented complements to university mathematics are needed. The emphasis, in the previous citation, on "coherence" and "quality assurance", still leaves much room to fill in, in relation to (3) and the more specific questions RQ1-RQ2: how, in fact, can well-acquired elements of "basic undergraduate mathematics" be developed and tuned towards the needs of the future teacher in a coherent way? After a brief discussion of the experimental context, we shall turn to this question while, as already mentioned, focusing on some central mathematical objects.

A specific institutional context

A considerable part of TEDS-M was focused on mapping out teacher education system at a global level, briefly explained above. We shall now delve further into local aspects related to RQ1-RQ2.

We consider these in the context of the largest mathematics programme in Denmark which offers teacher qualification for upper secondary school, offered at the University of Copenhagen. In Denmark, only upper secondary teachers receive their initial education in universities. After graduating from university, teachers have to pass a practical and theoretical course on pedagogy, while teaching; the subject specific parts of this course are quite limited, and as the various university programmes are quite different, the course has few if any concrete links to these.

From the list of courses listed by Schmidt et al. (2013), all of the general mathematics courses (and much more) are required for future teachers studying at the University of Copenhagen. Meanwhile, only two units specifically directed towards teachers are currently offered: a general course on didactics of mathematics labelled DidG (with some parts being shared with other science disciplines, due to the teachers having to specialise in two disciplines), and a course labelled UvMat (Mathematics in a teaching context). The first course corresponds roughly to the "math instruction" unit mentioned by Schmidt et al., while UvMat covers a relatively wide range of elementary school mathematics subjects (besides functions and equations, also number systems, discrete mathematics and statistics), all aiming at providing students with a deeper knowledge of these subjects in view of preparing them as future teachers with respects to how these domains appear in Danish upper secondary school.

Both DidG and UvMat deliberately draw on elements of the undergraduate courses, and thus aim at providing elements of the "higher standpoint" called for by Klein, as well as being capstone courses in the sense further described by Winsløw and Grønbæk (2014). The two courses are still quite different in the sense that DidG is focused on cases and methods of teaching, while UvMat is focused on mathematical content. Both courses involve (as other university courses) both lectures and extensive work with assignments or "exercises".

As in the study (Winsløw & Grønbæk, 2014) of "challenges" met by such a capstone course, we shall focus here on how UvMat attempts to tackle concrete instances of (3). In that paper, it was pointed out that UvMat does not attempt to address $R_S\left(t,O\right)$ directly, while students are still in position σ within *U*; this may also to some extent represent a difference with DidG. In our recent paper (Winsløw & Huo, 2023), we described a main strategy of the course as supporting students in a transition represented as

$$R_U\left(\sigma,\omega\right) \xrightarrow{T} R_U\left(\sigma,O\right)$$

through the design of tasks *T* that somehow link a university mathematical praxeology ω with a school mathematical praxeology *O*. As some of the university level praxeologies are also developed further within the course (rather than simply drawn from standard courses) a full representation of the course objectives is

(4) $$R_U(\sigma,\omega) \to R_U^*(\sigma,\omega) \xrightarrow{T} R_U(\sigma,O)$$

and with this extension, the course can be said to offer many concrete proposals related to RQ1-RQ2. In particular, the task design is used not only in the development but also in the assessment of $R_U^*(\sigma,\omega)$ and $R_U(\sigma,O)$, or combinations of these. We note that T itself does usually not belong to the types of tasks found in ω or O, but is designed to link these, while drawing on $R_U^*(\sigma,\omega)$ and enriching both this and $R_U(\sigma,O)$.

Local mathematical context: real numbers

In our recent research in the context of UvMat (within the frame of the second authors' thesis) we have focused on the students' knowledge about the system of *real numbers*. This system can roughly be described as a set, $\mathbb{R}$, equipped with arithmetical operations, an order structure, and a related topology. All of these are crucial to central domains of upper secondary mathematics, including calculus, analytic geometry, and vector algebra (over $\mathbb{R}$), among others. The real number system is of course linked to and based on subsystems, especially the systems of integers and of rational numbers. Nevertheless, there are considerable and general differences between how these number systems appear in university and school institutions. In this section we present these along with overall UvMat choices related to RQ2 (chiefly, at the level of theory).

Real numbers in undergraduate mathematics

Real numbers are especially fundamental to calculus and analysis, where university students will meet more or less deep treatment of some of their properties related to limits and more generally, order structure. Even at university, such properties may be simply claimed or presented as evident, especially in calculus courses. In more theoretical courses, students are presented with the notion of *Cauchy sequence*, and the fundamental property that Cauchy sequences of real numbers converge. Results related to limits and continuity of functions get a more rigorous foundation in axioms or claims about the real number system, and the related topology of the real number set. Also $\mathbb{R}$ itself may get to be defined in some way, most commonly as the completion of $\mathbb{Q}$. The latter number system—of rational numbers—is usually taken for granted in analysis courses, while a more formal treatment appears in abstract algebra (as the field of fraction determined by the integral domain $\mathbb{Z}$). However, as algebra and analysis courses operate independently in the undergraduate curriculum, these constructions will appear rather disconnected to students. Also, introductory analysis texts typically pass over the construction of $\mathbb{R}$, and present only some of the fundamental properties (like the supremum property) as an early stepping stone towards more technical results, such as the extremal value theorem for continuous functions on compact sets. Students are then exposed to a rapid succession of theorems and proofs, confirming and adding to what they learned in calculus at secondary or university level. Special functions (like exponential or trigonometric functions) still appear in examples, but they are (like the real numbers themselves) not treated any further. The end result, which we can write roughly as $R_U\left(\sigma,\mathbb{R}\right)$, is then what students retain from these various expositions to the properties of real numbers, mostly within calculus and analysis courses. Naturally, many other mathematical objects than numbers strictly speaking—such as functions, operations on functions and various results on these—contribute to students' theoretical and practical conception of the numbers. But about these in isolation, students may actually know little more than what they learned in school.

Real numbers at primary and secondary school

The real numbers appear little by little, and in a much more fragmented and intuitive way, in primary and secondary school mathematics (see e.g. González-Martín et al., 2013), both within arithmetic of natural numbers, integers, and rational numbers, and also in geometry, school algebra, and (based on these) early calculus.

The idea that each point on the "number line" correspond to a number, also appears early on, with the representations of integers and fractions helping to view these as related and subject to a common order. Since digital technologies play a great role in calculation with numbers, both in school and society, *decimal* representations of numbers are likely to occupy a strong place in the pupils' relationship to the mathematical object $\mathbb{R}$, that is $R_S\left(p,\mathbb{R}\right)$. Arithmetic operations are supported by handheld calculators from primary school on. Both these and the order structure are more straightforward with finite decimals than with fractions. Finite decimals also seem to exhaust the points that can be identified on a number line equipped with a scale or ruler.

The fact that fractions are needed both to define finite decimals, and that not all fractions correspond to finite decimals, is not really treated. Of course, periodic or otherwise strange "decimals" may be contended to be really somehow "infinite". If finite decimals are not carefully defined in terms of fractions, this new variety of decimals may also pass silently into $R_S\left(p,\mathbb{R}\right)$ as a fact of life which does not require further explanation or questioning. Indeed, students will encounter "numbers" like roots of integers or the mysterious fellow π that are chiefly "real" to them as a consequence of being easily manageable on a calculator (where they work, indeed, as and with decimals).

At more advanced points in upper secondary school, the work with functions and equations is also heavily supported by graphical representations and (at least to produce these) by digital tools. This will then add more geometrical or visual elements to $R_S\left(p,\mathbb{R}\right)$, mostly in a non-conflictual way: the intersections of curves can be both seen and calculated in consistent ways. Since the work is, at this point, also often heavily supported by algebra, many pupils struggle even when calculating tools are proposed as means to overcome some of the more technical points of the tasks they are assigned; but these tasks are frequently constructed so as to limit these struggles through the use of standard techniques. Pupils are rarely or never exposed to tasks that challenge their intuitive notion $R_S\left(p,\mathbb{R}\right)$ of the real numbers as points and decimals.

Real numbers in UvMat

We now present how UvMat addresses RQ2 at the level of mathematical theory on $\mathbb{R}$, in view of the discontinuities outlined in the preceding subsections. The main idea is to formalize the idea of real numbers as infinite decimals *on the basis of* theory from the undergraduate analysis courses.

During the fourth week lectures of UvMat, the construction of $\mathbb{R}$ as the completion of $\mathbb{Q}$ is rapidly reviewed and institutionalized, including the existence of suprema for non-empty subsets of $\mathbb{R}$ with an upper or lower bound (presented as an "axiom" in a prerequisite analysis course). The starting point is thus the existence of a complete ordered field $\mathbb{R}$ containing the integers. From the supremum axiom, we derive the Archimedean property: for every real number x there is a unique integer m , such that $m \leq x < m+1$.

From this point, the lectures follow Sultan and Artzt (2018, pp. 335-353) to show the existence of decimal representations of real numbers x, through an inductive construction of a sequence d_k of finite decimals such that $0 \leq \left|x - d_k\right| < 10^{-k}$ for all k. By the definition of limit, which is well known to the students, this means $x = \lim_{k\to\infty} d_k$. It is also shown that sequences of the form $d_k = \sum_{j=1}^{k} \frac{c_j}{10^j}$, where $c_j \in \left\{0,\ldots,9\right\}$, always converge, and that if two such sequences have the same limit—say $\sum_{j=1}^{\infty} \frac{b_j}{10^j} = \sum_{j=1}^{\infty} \frac{c_j}{10^j}$—then either $b_j = c_j$ for all j, or one of the sequences of decimals becomes eventually 0 (say, $c_j = 0$ for $j > N_0$) while the other becomes eventually 9 ($b_j = 9$ for $j > N_1$), and moreover if N is the least natural number that realizes both properties, we have $c_j = b_j$ for $1 \leq j < N$ and $c_N = b_N + 1$. This, with some minor details added, proves that real numbers *are* in fact "infinite decimals" $\sum_{j=1}^{\infty} \frac{c_j}{10^j}$ in the sense that all real numbers do have an infinite decimal representation, that every infinite decimal representation corresponds to a real number, and that this representation is unique *except* if it terminates with 0's or 9's (in which case there are exactly two such representations).

Naturally, other properties, such as the rational numbers being exactly all real numbers with an eventually periodic decimal representation, are also added (for some students recalled) to enrich this formalization. The lectures also address, briefly, whether $\mathbb{R}$ could be simply *defined* as the set of formal infinite decimals and point out some difficulties related to arithmetic operations.

Many students have certainly become aware—often in school—of facts like that finite decimals such as $1.02 = 1.02\overline{0}$ also have an alternative infinite decimal representation (here, $1.01\overline{9}$). But it is clearly new to them that they can be derived from university material on $\mathbb{R}$. We thus have a theoretical extension $R_U^*(\sigma,\mathbb{R})$ of $R_U(\sigma,\mathbb{R})$, which formalizes crucial elements of $R_S(p,\mathbb{R})$, with a potential of strengthening a future $R_S(t,\mathbb{R})$—at least in the sense of denaturalizing, for the teacher, the intuitive idea of infinite decimals, as a way to think of general real numbers. Also, crucial *practices* of the teacher—such as relating to the way computers handle real numbers—could be prepared by it, as we shall argue in the next section, when considering some elements of the tasks students engage in to build $R_U(\sigma,O)$.

A case of task design in the local context

University mathematics courses (such as UvMat) present students with some praxeological elements of a more theoretical nature during lectures, while devolving assignments and other tasks to students in view of strengthen their relationship with the praxeology at large. Especially in more theoretical courses, one may seek to engage students in tasks which build or extend theory (e.g. Grønbæk & Winsløw, 2007) and UvMat does so in at least through mandatory weekly assignments which develop some theoretical point, often starting from examples. They can be considered concrete proposals for the aims explicit in RQ1: build new, school relevant relationships of type $R_U(\sigma,O)$ while drawing on $R_U(\sigma,\omega)$ or a possible extension $R_U^*(\sigma,\omega)$. As RQ1 suggests, the design work departs from some school relevant $R_U(\sigma,O)$, and seeks a relevant $R_U(\sigma,\omega)$ or $R_U^*(\sigma,\omega)$ that could be used to build $R_U(\sigma,O)$.

Among the crucial new objects introduced in upper secondary school are exponential, logarithmic and trigonometric functions, whose importance in mathematics and other disciplines need no defence. We wish to strengthen $R_U(\sigma,O)$ related to these, while drawing on the extension $R_U^*(\sigma,\mathbb{R})$ outlined above. In particular we consider that knowing an algorithm which computes a function "from the decimals of the input to the decimals of the output" could reinforce the students' relationship to the (school) model of the real numbers and relate it to a non-trivial function. We shall now consider a proposal for how to do so in the case of logarithms.

An algorithmic approach to logarithms

The assignment is based on an algorithm proposed by Goldberg (2006) for the computation of logarithms "digit by digit". The algorithm is most easily introduced by way of an example.

Consider $x = 432.1$; the idea to compute $\log_{10} x$ is to determine the decimals of a real number $y = N.d_1 d_2 \cdots$ satisfying $10^y = x$. We should thus have

$$(*) \qquad 432.1 = 10^{N + \frac{d_1}{10} + \ldots} = 10^N \cdot 10^{\frac{d_1}{10} + \ldots} = 10^N \cdot m_1$$

where $m_1 = 10^{0.d_1 d_2 \cdots}$. As $1 \leq m_1 < 10$, the right-hand side has $N+1$ digits before the comma, so from the left-hand side we get $N = 2$. Dividing (*) by 10^N we get

$$4.321 = 10^{0.d_1 d_2 \cdots}$$

and as we now wish to proceed to determine d_1 we rewrite this as

$$4.321^{10} = (10^{0.d_1 d_2 \cdots})^{10} = 10^{d_1 . d_2 d_3 \cdots} = 10^{d_1} \cdot m_2 \cdot$$

Again $1 \leq m_2 = 10^{0.d_2 d_3 \cdots} < 10$ so d_1 is one less than the number of digits in the integer part of $4.321^{10} \approx 2269042.7$, that is, $d_1 = 6$. Note here that the computation of 4.321^{10} requires nothing more than basic multiplication and results in a finite decimal. We continue with

$$\left(2.269042671\right)^{10} = 10^{d_2} \cdot m_3$$

to find d_2, and so on.

With this procedure, all it takes to compute $\log_{10} x$ is to be able to count the number of digits in the integer part of a given number and multiply the number by itself (10 times). It can thus—for a given *finite* decimal—be done with only the four basic operations. In particular exponential functions are not required to carry out the algorithm. Of course, such knowledge is required to *verify* that it computes an inverse of $x \mapsto 10^x$ – or as above, to develop the algorithm for this purpose.

In the assignment developed for the UvMat students we chose to focus on the verification issue, and on the possibility of computer implementation. Details are given in the next section. The motivation for these choices was that the detailed construction and properties of exponential functions were already treated in the lectures, with only brief remarks about logarithms as the inverses of these (cf. Winsløw, 2013). This builds only on $R_U\left(\sigma, \mathbb{R}\right)$ from university courses, and not on $R_U^*\left(\sigma, \mathbb{R}\right)$ more directly connected to school models of $\mathbb{R}$. However, as students always compute and graph transcendent functions with digital devices, the algorithmic or decimal approach, along with computer experiments, explains what that "black box" may contain. At a deeper level, it formalises the intuitive idea of $\log_{10} x$ as "the number of times 10 divides x" (Weber, 2016).

Goldberg (2006) developed the same ideas with other bases (both for the logarithms and the representation of real numbers); indeed, computation is simpler in base 2. In the course we did not include expansions of real numbers in bases other than 10, as this further extension of $R_U^*(\sigma,\mathbb{R})$ has much less relevance to $R_S(t,\mathbb{R})$ than the formalisation of decimal expansion.

An example of a student assignment

The assignment begins with a preamble, explaining its purpose as "developing a method to compute $\log_{10} M$ for a given $M > 0$ (...) in the sense that we compute the decimals of $\log_{10} M$ successively, using only basic arithmetic operations."The assignment had six tasks:

a) Prove from properties of $y \mapsto 10^y$ that for any $x > 0$ there is a unique $y \in \mathbb{R}$ so that $10^y = x$. If we have a method to compute such y for any $x > 1$, how can we do it for $0 < x < 1$?

b) Assuming $x > 1$, show existence and uniqueness of $c(x) \in \mathbb{N} \cup \{0\}$ such that $10^{c(x)} \leq x < 10^{c(x)+1}$.

c) Explain how to determine $c(x)$ from the decimal representation of x. Give a couple of examples.

d) Given $x > 1$ and letting y be as in a), we wish to find the decimal representation $y_0 + \sum_k y_k 10^{-k}$ of y. Show that this can be done by : $y_k = c(x_k)$ when we define, recursively: $x_0 = x$ and $x_k = \left(\frac{x_{k-1}}{10^{c(x_{k-1})}}\right)^{10}$ for $k \in \mathbb{N}$.

e) Use d) to compute $\log_{10} 57.64$ with four decimals.

f) Interpret a given *Maple* routine as implementing d).

The aim of this assignment is primarily that students work on a non-trivial case of an algorithm that computes the values of a function "digit by digit" and could therefore be thought of as the mathematical basis of a "calculator button" (or command) to compute that function. In other words, the primary point is more on the decimal representation of real numbers, and less on the concrete example ($\log_{10}$).

In task a), students need to use the property that $y \mapsto 10^y$ is a bijection from $\mathbb{R}$ to $\mathbb{R}^+$. The rest of the assignment is about the algorithm to actually compute (the decimals of) y for a given x. Tasks b) and c) were designed to define the auxiliary function $c : [1,\infty[\to \mathbb{N} \cup \{0\}$

which is central to the algorithm. In d) students must then explain how the given algorithm allows us to find the decimals of y (from x). The task e) allows students to try out the algorithm on a concrete number (like we did above) and task f) provides students with a piece of code which they should recognise as implementing the algorithm from d), and try out in Maple.

Now, we consider briefly how students draw on $R_U\left(\sigma,\mathbb{R}\right)$ to answer the assignment, based on the students' answer sheets and interviews conducted with them in view of gaining further insight into what they learned from working with the assignment.

In task b), there was a considerable diversity among student answers. More than half of students considered that what is to be proved is equivalent to existence and uniqueness of $c\left(x\right)$ such that $c\left(x\right)\leq y<c\left(x\right)+1$, where $x=10^y$, but without explicitly referring to the result from a). Moreover, the existence of the "integer part" is treated as obvious by students, while in the course it was proved to be a consequence of the Archimedean property of $\mathbb{R}$. So, at this point students continue to treat properties of $\mathbb{R}$ with the same informality as is usual in high school.

Another way some students take is to divide $\left[1,\infty\right[$ into segments $\left[10^k,10^{k+1}\right[$, observing that $\left[10^k,10^{k+1}\right[$ and $\left[10^{k+1},10^{k+2}\right[$ are disjoint for all $k\in\mathbb{N}\cup\left\{0\right\}$. Therefore, as students explained: "It is then true that the union of all these disjoint sets corresponds to $\left[1,\infty\right[$, and it is then true that a number will always lie in just one of the sets". Some students used a proof by contradiction to show that a given x could only be in one interval of type $\left[10^k,10^{k+1}\right[$. These arguments are more similar to what students will have met at university, relying explicitly or at least implicitly on the equality $\left[1,\infty\right[=\bigcup_{k=0}^{\infty}\left[10^k,10^{k+1}\right[$.

In task c), students learnt how to determine $c\left(x\right)$ by only looking at the decimal representation. It was not difficult for students to find that $c\left(x\right)$ is the number of digits in the integer part of x minus one. There were, curiously, still some groups who did not give any examples, as asked for by the task. This task is used to help students to solve the core part—task d).

In task d), students were asked to explain the algorithm, where one really needs to use explicitly that a decimal representation of a real number is a kind of sum, and also the property $10^{a+b}=10^a10^b$. However, some students mixed formal and informal representations. For example, one group used $y=\lfloor y\rfloor+0.c_1c_2c_3\ldots$ ($\lfloor y\rfloor$ is the integer part of y) as the representation of infinite decimals when they were solving the task although y is represented as $y_0+\sum_k y_k10^{-k}$ in the task. The informal representation is of course closer to high school practice. To explain infinite decimals as infinite series is on the other hand a main point in this part of the course. Although those informal expressions did not affect the essence of students' final proof, we still observe that some students are somewhat limited by high school conceptions when faced with common high school notions. Their reluctance to use formal reasoning in relation to such notions is a very general experience in the course.

Task e) asked students to try out the algorithm with a concrete example and all students succeeded by following the steps in task d). Task f) tested whether students could relate the code to their own explanation in d). These two tasks are follow-up questions to the task d) which are hoped to increase the students' grasp of the point of the assignment.

We interviewed 8 students after they got the revision comments, mainly to learn what they saw as the point of this assignment. All students agreed that the assignment showed them another way to calculate logarithms where they got new insight into logarithms, beyond or behind its status as a "button" on calculators. One student described "*…I knew that the logarithm was the inverse to the exponential but I never quite figured out how to calculate them. But now we learned a little bit about that with this approximation and then of course something about how maple works…*" Some students also felt it was very surprising that they actually could calculate logarithms by hand: "*I think I learned how to easily calculate logarithms by hand without using Maple.*"Most students did not focus on the relation between the infinite decimal representation of real numbers and this assignment, even though this was the main focus in that course week. Only one interviewee talked about the computation of decimals "digit by digit": *I think it is to develop a method to actually calculate the logarithms sequentially one decimal at the time.*

The mixed student impression of what this task was for, illustrates a general challenge with assignments in the course, namely that students may succeed with carrying out certain technical steps (drawing on some $R_U(\sigma,\omega)$) without seeing how the steps, together, support a major point in relation to high school mathematics. It visibly does not suffice to state the overall point in a preamble. One point that needs more attention is how to formulate "summary questions" which allow students to reflect on more general points of the assignment, without these questions being perceived as of the type "write your opinion" (or worse, "guess what the teacher wants"). In this case, the meaning of an explicit or computational specification of a function is extended from "algebraic formulae" (thoroughly known from school) to a recursive algorithm that makes explicitly use of the decimal representation of real numbers. Another possibility is to institutionalise such theoretical points in a follow-up lecture, referring explicitly to the assignment. Naturally, the whole set-up with lectures and exercises could be questioned, however in a relatively traditional institutional context, there are also strong conveniences by keeping the formats that students are used to.

Conclusion

It is interesting to observe that while modern research into the nature and effects of mathematics teacher knowledge has adopted relatively global categories and viewpoints—corresponding to what might be represented as $R_S\left(t, M\right)$ where M is in some sense "school mathematics"—the original point of view of Klein was much more local, considering for instance how future teachers' relationship $R_U\left(\sigma, \mathbb{R}\right)$ to the real numbers could be developed based on the "advanced standpoint", of type $R_U\left(\sigma, \omega\right)$, developed at university. The global viewpoint is certainly important when considering policy issues related to institutions and international comparison, which in some cases even goes beyond considering the single school discipline $M = \cup_i O_i$. Still, the more local viewpoint needs to be recovered in order to address the didactical question of how to actually develop and assess relationships of type $R_U\left(\sigma, O\ \right)$, while drawing on some $R_U\left(\sigma, \omega\right)$. Even some policy issues—like what contents to include or reinforce in study programmes for future teachers—depends on what we know at this level.

Klein's concrete proposals to this end were given in the form of notes from a lecture course. In this paper, we have developed and exemplified an alternative and altogether more student-oriented approach related to task design. At the same time, we have exemplified the general scheme (4): with $R_U\left(\sigma, \omega\right)$ being given by an undergraduate mathematics programme that is not specifically designed for teacher education, it may be necessary to develop such relationships further to what we have denoted $R_U^*\left(\sigma, \omega\right)$, in order to create viable tasks that can lead students to didactically relevant new relationships to school praxeologies, such as a deeper understanding of decimal representations of real numbers, special functions and so on. We emphasize that developing such tasks requires simultaneous and up-dated knowledge of both the undergraduate prerequisites and high school mathematics. Moreover, previous studies of how pupils and teachers at large relate to a given high school praxeology could be invested in the selection of problematic local contexts and in the design process. Other simple aims and methods that one can pursue in the design of such tasks were proposed by Huo and Winsløw (2023).

We do not claim that task design is the only or even a sufficient means to achieve, for instance, a relationship to the real numbers which is relevant to how these appear at secondary level, and in other contexts where digital tools are more dominant than in scholarly mathematics. In fact, our case also suggests that just like regular undergraduate courses, capstone courses may benefit from a vigorous dynamic between students' work with challenging aspects of high school mathematics and lectures which focus on extending deepening their theoretical knowledge in directions that are relevant to such student work. Further research is needed to estimate the effects of such courses on actual relations of type $R_S\left(t, O\right)$, and effects of $R_S\left(t, O\right)$ on the relationship to O of the students of t.

Thus, from a modern point of view—where the gap between the standard undergraduate mathematics programme and mathematics in secondary school and society has certainly increased—RQ1 cannot be seriously considered without also taking RQ2 into account. In the case considered, the rigorous approach to infinite decimals requires revisiting and extending previous work on properties related to completeness. In many contexts, identifying such needs could lead to renegotiating key elements of the external didactic transposition at university, with the possibility of enriching the general undergraduate programme.

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