Pre-service teachers’ classroom stagings of proving-related activities and possible effects of Klein’s discontinuity

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Proof is a core element of mathematics. It therefore plays an essential role in university mathematics studies and thus also in the training of pre-service mathematics teachers. However, it has been shown that in-service teachers face problems in adequately implementing proof in their teaching. We consider this break between university studies and professional practice as part of what Klein called the “second discontinuity”. Employing an activity-theoretical framework, we investigate in this paper the question of how lesson stagings of pre-service teachers at the end of their study program indicate systematic difficulties in staging proving-related activity for the classroom and how discontinuity experiences triggered by differences between proving-related practices in school and university could be considered as a possible cause for that. We present results from an observational study with pre-service teachers that reveal specific patterns in their proof-related behavior when planning and delivering lessons, and relate these back to sources for possible discontinuity experiences of pre-service teachers. As a practical implication, our results provide indications of how the topic of proof could be targeted in mathematics didactic teacher education.

Keywords: proof, activity theory, classroom stagings, pre-service teachers, argumentation and proof

Bauer, T. and Müller-Hill, E. (2025). Pre-service teachers’ classroom stagings of proving-related activities and possible effects of Klein’s discontinuity. *Recherches en didactique des mathématiques*. 45 (3), 205-237, https://doi.org/10.46298/rdm.15555,

Mise en scène par les enseignants en formation initiale des activités liées à la démonstration et effets possibles de la discontinuité de Klein

La preuve est un élément central des mathématiques. Elle joue donc un rôle essentiel dans les études universitaires de mathématiques et, par conséquent, dans la formation des enseignants de mathématiques en formation initiale. Cependant, il a été démontré que les enseignants en exercice rencontrent des problèmes pour mettre en œuvre la preuve et la démonstration de manière adéquate dans leur enseignement. Nous considérons que cette rupture entre les études universitaires et la pratique professionnelle fait partie de ce que Klein a appelé la « deuxième discontinuité ». En utilisant un cadre théorique de l'activité, nous étudions dans cet article la question de savoir comment les mises en scène des leçons des enseignants en formation à la fin de leur programme d'études indiquent des difficultés systématiques dans la mise en scène d'activités liées à la preuve et la démonstration pour la classe et comment les expériences de discontinuité déclenchées par les différences entre les pratiques à l'école et à l'université pourraient être considérées comme une cause possible de ce phénomène. Nous présentons les résultats d'une étude d'observation menée auprès d'enseignants en formation qui révèlent des schémas spécifiques dans leur comportement lié à la preuve et la démonstration lorsqu'ils planifient et dispensent leurs cours, et nous les mettons en relation avec les sources d'éventuelles expériences de discontinuité des enseignants en formation. D'un point de vue pratique, nos résultats fournissent des indications sur la manière dont le thème de la preuve et de la démonstration pourrait être ciblé dans la formation des enseignants en didactique des mathématiques.

Mots-clés : démonstration, théorie de l'activité, mises en scène en classe, enseignants en formation initiale, argumentation et démonstration

Implementaciones en el aula de actividades relacionadas con la demostración por parte de profesores en formación y posibles efectos de la discontinuidad de Klein

La demostración es un elemento esencial de las matemáticas. Por lo tanto, desempeña un papel esencial en los estudios universitarios de matemáticas y también en la formación de los profesores de matemáticas en formación inicial. Sin embargo, se ha demostrado que los profesores en activo tienen problemas para implementar adecuadamente la demostración en su tarea docente. Consideramos esta ruptura entre los estudios universitarios y la práctica profesional como parte de lo que Klein denominó la «segunda discontinuidad». Empleando el marco teórico de la actividad, en este artículo analizamos la cuestión de cómo la implementación de las sesiones de los profesores en formación al final de su programa de estudios indica dificultades sistemáticas en la actividad relacionada con la demostración en el aula y cómo las experiencias de discontinuidad provocadas por las diferencias entre las prácticas relacionadas con la demostración en la escuela y la universidad podrían considerarse como una posible causa de ello. Presentamos los resultados de un estudio observacional con profesores en formación que revelan patrones específicos en su comportamiento relacionado con la demostración cuando planifican e imparten clases, y los relacionamos con las fuentes de posibles experiencias de discontinuidad de los profesores en formación. Como implicación práctica, nuestros resultados proporcionan indicaciones de cómo podría enfocarse el tema de la demostración en la formación didáctica del profesorado de matemáticas.

Palabras claves: demostración, teoría de la actividad, implementaciones en el aula, profesores en formación, argumentación y demostración

1 Introduction

Pre-service teachers (PSTs) in university mathematics courses experience mathematics as a proving science: mathematical proofs play a central role in lectures as well as in exercises and exams. Hardly any mathematics lecture simply communicates mathematical theorems without proofs – the question of "why" and the desire for deductive chains of reasoning as justifications and explanations are part of the very essence of the subject. Even though the notion of proof is a complex and context-sensitive construct (see Hanna & De Villiers, 2008, 2012), instructors of university mathematics courses, at least implicitly, expect PSTs to develop through this lived practice a disposition to act in a proof-oriented way. If we use the term *disposition* in the sense of Perkins et al. (1993), that is, as a triad of inclination, sensitivity, and ability, then we could describe this expected disposition as the following triad: a tendency to tackle mathematical work in ways that contribute to answer a certain type of why-question, an alertness to occasions that are appropriate to engage in such ways, and the ability to follow through with respective proving-related strategies and techniques. In the case of pre-service teachers, it is hoped that this disposition will also manifest itself in an appropriate way throughout their later teaching. Schoenfeld (2020), for example, indirectly articulates this hope when he calls for mathematics instruction that allows PSTs to experience mathematics as "a discipline of exploration and sense making" that is about "seeing how and why things fit together the way(s) they do" (p. 1169).

Klein (1932) already expressed general doubts that the transition from university education to teaching in schools would be so smooth: he coined the term of "double discontinuity" and strongly pointed out that in many cases teachers resort to long-established teaching traditions instead of using knowledge learned at university. Recent research on how teachers deal with proof largely confirms such doubts (Ko, 2010; Bieda, 2010; Buchbinder & McCrone, 2020, see also Wasserman et al., 2023) and shows that teachers often experience strikingly strong breaks between proving-related activities and actions at school and at university. When it comes to proving-related action in the classroom, the teachers' main focus is often on situations that are explicitly designated for proof, such as "proof lessons" on the Pythagorean Theorem or the Thales Theorem. To be able to design and deliver such lessons effectively is indeed very important and in itself a demanding task. From our point of view, however, activities that are related to proving (such as finding arguments, conjecturing, explaining the "why") should not occur only in isolated places in mathematics instruction. Therefore, we find it equally important to ask whether teachers develop a proof-oriented action disposition that will manifest itself in their teaching throughout, that is, also outside of "proof lessons", in a way that conveys an authentic image of mathematical thinking to their students and engages them in activities that can "support the development of productive mathematical identities" (Schoenfeld, 2020, p. 1171).

In this paper we strive for a more nuanced view of what might be "discontinuities" related to proving. To this end, we investigate how lesson stagings of pre-service teachers at the end of their study program allow to trace the presence of a proof-oriented action disposition and to explain observed insufficiencies as a result of discontinuity experiences. Concretely, we seek to provide a phenomenology of observed teaching behavior and explain it as a result of differences between proving-related practices in school and in university that PSTs may have experienced. As stated above, we posit that teachers should display a disposition for proving-related action not only when mathematical situations are explicitly labeled as "proof" (by curricula and textbooks), but by their ability to make competent use of opportunities for proving-related activities (such as making and exploring conjectures, finding and discussing arguments) in their mathematics teaching throughout. Therefore, we deliberately focus observational attention on situations in which PSTs have not been explicitly assigned to teach a "proof lesson", but instead observe their actions when teaching self-selected lesson topics. Our study is based on data from a capstone course with prospective teachers (first results were reported in Bauer & Müller-Hill, 2022a). We employ activity theory as a conceptual, reconstructive framework for analysis and as an explanatory theory.

2 Theoretical background

2.1 The second discontinuity

The term "double discontinuity" characterizes a complex problem in grammar school teacher education, which Klein (1932) pointed out impressively. It contains several sub-problems and conjectured cause-effect-relationships. According to Klein, PSTs entering university are "confronted with problems which did not suggest, in any particular, the things with which he had been concerned at school". Later, after graduation, a PST will "scarcely [be] able, unaided, to discern any connection between this task and his university mathematics". Therefore, as Klein concludes, university mathematics will have "no influence on his teaching", and they will instead rely on old traditions of teaching (Klein, 1932, p. 1). Over time, authors have interpreted this statement in different ways and focused on different sub-aspects: On the one hand, it is used holistically for issues in teacher education that lie within the problem area addressed by Klein (e.g., Hefendehl-Hebeker, 2013). More specifically, it is often used in the sense of gaps that PSTs experience, distinguishing a first discontinuity (upon entering university studies) and a second discontinuity (upon graduation). Eichler and Isaev (2023) understand the two discontinuities as beliefs that PSTs have about the coherence between school and university mathematics (first discontinuity) and about the relevance that they see university mathematics as having for school mathematics (second discontinuity). Allmendinger et al. (2023), on the other hand, focus on abilities: they point out that double discontinuity in difference-oriented frameworks is interpreted as "PSTs' inability to connect pieces of mathematical content", and they emphasize that Klein, read literally, speaks rather of the connection between university mathematics and the task of teaching school mathematics. We refer the reader to the article by Wasserman et al. (2023), which provides a survey of recent related literature.

In this article, we will use the term "discontinuity" in two specific ways, speaking first of *discontinuities between practices* at school and university, and second of *discontinuity experiences* that PSTs make. To explain the former, we start with Klein: In his discussion he addresses, on the one hand, content-related problems when he says, for instance, that in the field of real analysis "the discontinuity between school and university […] is greatest" (Klein, 1932, p. 236). On the other hand, he also considers the "manner of instruction" that he characterizes as "intuitive and genetic" in school and as "logical and systematic" at the university (Klein, 1932, p. 6; c.f. Allmendinger 2016, p. 214). Such differences at the secondary-tertiary transition have been studied intensively in recent times, pointing out, for example, different modes of thinking and different organization of knowledge in school and university (see Gueudet, 2008, 2023). When referring to such differences, we will speak of *discontinuities between practices* at school and university. Clearly, not all school classrooms are the same and not all university courses are the same, so PSTs have individual experiences that may differ from one another depending on the classes and courses they attended. We will use the term *discontinuity experiences* to refer to the individual experiences of gaps and disconnect that PSTs make between practices they encountered at school and university.

2.2 Activity theory

In our approach, we build upon activity theory (AT) in the version established by Leontjew (1978, 1981) and developed further within didactical and pedagogical discourse in Germany, for example by Giest and Lompscher (2006), as well as within international scientific discourse dealing with the development and interaction of mind and sociocultural environment (see, e.g., Kaptelinin, 2005). We work with Leontjew's initial version of AT instead of the one developed by Engeström (1987, 2001), as we are concerned with the (socially mediated) individual development of PSTs rather than with organizational change (see, e.g., Hershkowitz & Schwarz, 1999, p. 66, Chaiklin, 2019, p. 15; for a focused delineation of Leontjew's and Engeström's work in this regard see Cong-Lem, 2022).

According to AT, human activities principally aim at changing and transforming a shared reality (that is accessible to our senses or socially constructed) in order to to fulfill certain needs. AT distinguishes three constitutive elements of an activity: the impelling *motive,* the directing *target object* (see Kaptelinin, 2005), and *ways and means of action*. The motive to fulfill a specific need drives the activity towards a target object. The target object is the intended target state of transformation. According to Kaptelinin (2005, p. 5) it has a dual status: The target object "is both a projection of human mind onto the [...] world and a projection of the world onto human mind". This means that it is part of individual activity to see what could be an appropriate and reachable target object in a given situation. The motive as a driving force together with the target object as a "sense-maker" (Kaptelinin, 2005, p. 5) make the activity meaningful for the individual (Chaiklin, 2019, p. 12, 21). In order to produce the target object of an activity, the individual needs suitable ways and means of transformational action.

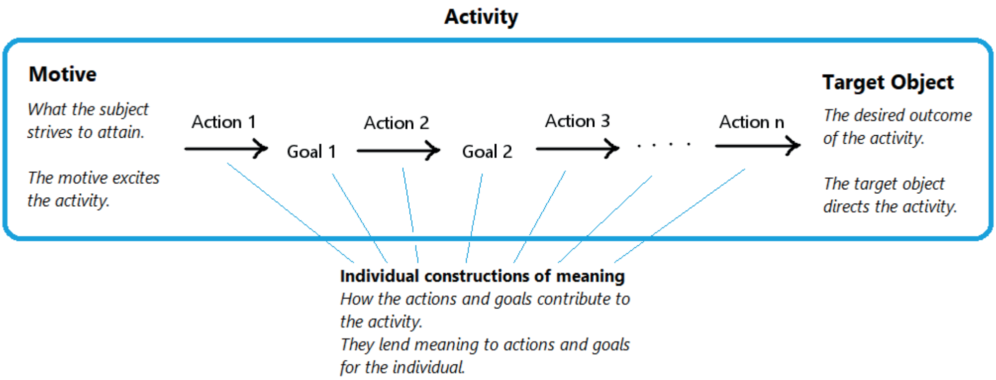

Figure 1. – Interplay of constitutive elements and constructions of meaning in individual activity.

AT, as formulated by Leontjew, includes a developmental perspective that considers both individual, psychic development and cultural-historical development, and advocates that (collective and individual) human activity develops in a way that is influenced by socialization and the development of language, as well as by division of labor and development of tools. Cultural-historical development can lead to the development of complex activities in which the motive is not realized through a single action, but through a sequence of partial actions, including automated operations (Cong-Lem, 2022, p. 1098), whose goals contribute suitably to establishing the target object. Fig. 1 shows the structure of a complex activity, with a sequence of actions and goals. We will use this structure in our data evaluation (Step 1 in Sect. 3.2).

AT systematically integrates the possibility that *alienation* can occur, which means that in the midst of complex activities, the acting individual only perceives the goals of single actions, but not the motive and target object of the activity as a whole. This could happen, for example, if a division of labor is practiced in a collective activity or if the activity is scaffolded in small steps (as can occur in learning scenarios). There is then the possibility that the single actions are not meaningful for the individual (thus the individual is not performing an activity in the sense of AT), even though the single actions may contribute to the target object. This applies in particular to highly developed mathematical activity, such as activities related to mathematical proving, since they are complex activities of that kind. To mitigate alienation, individuals need *constructions of meaning*, which mediate between single actions and motive/target. The term "construction of meaning" captures two aspects: on the one hand, an action goal can have personal relevance for an individual (which may ground a tendency to act accordingly) (see Suriakumaran et al., 2017) and on the other hand the individual can recognize the "objective sense intended" (Howson, 2005, p. 18), that is, the objective contribution that the action makes to achieving the target object.

Summing up, AT provides the following crucial insights (we apply them in Step 4 of our data evaluation, see Sect. 3.2): The extent to which an individual can perform and perceive a sequence of goal-oriented actions and operations as parts of meaningful (individual) activity depends

- firstly, on the individual and situationally available ways and means of action (e.g., individual abilities), because the individual needs them in order to set goals and carry out actions,
- secondly, on whether the individual has developed effective constructions of meaning as mediators between action goals, motive and target object.

While these aspects concern PSTs and pupils as *learners* of mathematics, PSTs also need to develop constructions of meaning for their role as *teachers* of mathematics, serving as a basis for staging mathematical activities in class. According to AT, these constructions of meaning need to mediate in multiple ways: between action goals, motive and target object, between their occurrence in school and university mathematical practices, between disciplinary and learning/teaching practices. Such constructions of meaning emerge from sufficiently rich, conscious experience and explicit and interconnecting reflection of mathematical activity as part of lived school and university practices in which the individual PST participates. We employ this insight explicitly in our AT-based explanatory principle stated in Sect. 4.2.2. AT supports this view with the "key assumptions in a psychological concept of activity" (the "sociohistorical assumption" and the "social assumption", Chaiklin, 2019, p. 11 ff.) as well as by the fundamental claims of AT that "activity is the minimal meaningful context for understanding individual actions" and that "consciousness stems from practice" (Hershkowitz & Schwarz, 1999, p. 66).

2.3 Proving-related activities and actions

In our study, we focus on proving-related activities and actions. Based on our understanding of a proof-oriented action disposition (see Sect. 1) and the AT concept of "activity", we conceptualize them as activities and actions that are directed towards producing (or contributing to produce) answers to a certain type of why-questions: why a mathematical proposition holds, why a specific mathematical phenomenon appears under specific conditions, why a certain mathematical concept is applicable in a given situation, why a certain mathematical object exemplifies the concept, or why a certain mathematical procedure works in the intended way. Answers to this type of question typically refer to the conceptual core of the subject matter and show the decisive reasons (relative to a particular background theory) for a statement being true or a concept being applicable. We use the term "proving-related" in a broad way to refer not only to activities and actions that lead to the production of final proofs in an accepted format as target objects, but also to activities and actions that lead to generalizing patterns (Bieda, 2010), as they occur in preceding stages (see Boero, 1999), such as exploring a problem situation, making conjectures, exploring them, and finding suitable arguments. This includes non-deductive forms of mathematical reasoning, for example inductive or abductive argumentation based on the variation of example cases, which play important roles typically assigned to mathematical proof (see Müller-Hill, 2019), such as explaining, discovering, convincing, communicating (Hanna, 2000).

2.4 Research questions

As stated in the introduction, we study how lesson stagings of pre-service teachers at the end of their study program allow us to trace a manifestation of a proof-oriented action disposition. We are aware that PSTs might intentionally not focus on proving-related activities in certain lessons (or parts of lessons) for well-chosen reasons. Therefore, we only consider those situations that they explicitly mark themselves as proving-related (we specify in Sect. 3.2, Step 0, how "marking" is defined). For each of these situations, we then analyze whether under an AT perspective a proving-related activity can be reconstructed in the situation. Our first research question can thus be formulated as follows:

(RQ1) In PSTs' lessons stagings, what kind of situations occur that the PSTs mark as proving-related but that the researchers cannot reconstruct as proving-related activities in an AT analysis?

We consider RQ1 to be relevant in two respects: First, if a proving-related activity cannot be reconstructed, then learners might not perceive a sequence of actions as parts of a meaningful individual activity. From an AT perspective, however, that perception is crucial for learners to build an authentic picture of mathematical thinking (or a "productive mathematical identity", as Schoenfeld, 2020, p. 1171, puts it). Second, these situations show a disconnect on the part of the PSTs between their stated intention and the execution. From an AT perspective, such a disconnect could indicate that the PSTs have difficulties in mediating between action goals and motive/target (and this points to possible issues in the development of constructions of meaning, see Sect. 2.2).

Our AT analysis in terms of motive, target object, and (sequences of) contributing actions and goals, enables us to identify patterns and phenomena that occur in these situations. In answering RQ1, we provide a phenomenology of these patterns and phenomena. We consider this as an appropriate basis for exploring possible explanations in terms of discontinuities that PSTs may have experienced between school-mathematical and university-mathematical practices. With these explanations we strive for a more nuanced view of what might be "discontinuities" related to proving:

(RQ2) How can these patterns and phenomena be explained in the AT framework as possible effects of discontinuity experiences that PSTs may have made?

We recognize that the two research questions may seem deficit-oriented at first glance, but identifying difficulties is, in our view an important first step in designing learning opportunities for PSTs that can address these issues in order to enable PSTs to bring proving-related activities to bear appropriately in their future teaching (see Sect. 5).

3 Methodology

3.1 Data collection

3.1.1 Sample

We collected data in a capstone course for PSTs in the summer semester 2021 at a medium-sized German university, which was taught by the second-named author and attended by the first-named author as an observer. The course was taken by PSTs in their final semester and is part of the mathematics didactics part of the teacher training for the grammar school teaching profession and for the teaching profession at middle schools. Prior to this course, PSTs complete in the field of mathematics didactics an introductory course on foundational themes and on theoretical basics of lesson planning, a school-based practical exercise in which, under the guidance of a mathematics teacher, lessons are planned by a small group of PSTs, delivered by one of them, and observed and reflected upon by the group, and a 9-week school internship. Also, they complete at least one specialized module (media in mathematics classroom, mathematical learning tasks), and at least one interface module on school mathematics from a higher standpoint (arithmetic/algebra, analysis, or stochastics).

The course was attended by 12 PSTs. A total of 8 sessions of 90 minutes each took place, of which the first session was reserved for an introduction and the last for a final evaluation. For each of the middle 6 sessions, a tandem of PSTs was assigned to prepare a lesson of 45 minutes on a specific teaching topic of their own choosing (and using literature of their own choosing). The specific topics had to be subsumable under one of two main topics that had collectively been chosen by all participants before the start of the seminar (see Table 1). We will refer to these lessons as *stagings* in the following. The stagings were conducted as teaching experiments with the other seminar participants as addressees and peer experts. In a short presentation of 15 minutes prior to the staging, the teaching tandem had to present key points from a longitudinal (across the grade levels) and cross-sectional (across topics within the selected grade level) didactic subject analysis of their chosen topic (thereby concretizing basic mathematical experiences ("Grunderfahrungen", Winter, 1995), basic mental models ("Grundvorstellungen", vom Hofe, 1995; vom Hofe & Blum, 2016), and guiding ideas with respect to their chosen topic and staging considerations). Furthermore, they display their application of didactical theories and concepts to identify possible staging and learning obstacles and justify didactical choices made in their planning. In addition, they presented their learning goals and the lesson plan for the staging. After the staging they received feedback from their peers and the course instructor. PSTs described and reflected their lesson planning and staging in detail in writing as part of a term paper.

The freedom PSTs had in choosing a specific teaching topic and in planning an associated lesson corresponds with our goal of investigating PSTs' behavior when they are not placed in the situation of planning a staging pre-declared as a "proof lesson". Table 1 shows the chosen topics for the six sessions (numbered A1, …, A6 in the following).

Table 1. Specific teaching topics chosen by the PST tandems

	Chosen specific topic	Topic area with respect to school curricula	Collectively chosen main topic	Grade
A1	area of right-angled triangles	planimetry	functions	6
A2	odd power functions	functions	functions	9
A3	half-life	exponential and logarithmic functions	functions	10
A4	zeros of functions	investigation of functions and their graphs	functions	11/12
A5	binomial formulas	working with variables	terms and equations	8
A6	scalar product	vectors and matrices	terms and equations	11/12

3.1.2 Observational data from PSTs' stagings

Since the sessions took place digitally due to the Covid19 pandemic, we could not video- or audiotape them according to local university privacy policies. Instead, the two authors observed the PSTs' stagings independently of each other and made notes on their observations. In accordance with the research questions, the observation was aimed at describing staging situations that are marked as proving-related by the PST (in the sense defined below in Sect. 3.2, Step 0). As an observation instrument we developed and used a semi-formalized observational sheet in which utterances and actions of the teaching PSTs were recorded. A comment column was optionally used for actions of the learners. For each situation or sequence that was marked as proving-related, brief descriptions or live transcriptions were documented in the sheet. The table header of the sheet is shown in Table 3; the following codes are used in the fourth column to preliminarily categorize the descriptions:

- G = goals (and motives, if any) formulated by the PST (indicated by corresponding wording such as "Our aim in this lesson/task is …")
- P = PST's prompts to the learners
- A = PST's actions (other than prompting and settings goals)

Time and phase of the lesson (in accordance with the lesson plan provided by the teaching PSTs in advance) or phase transitions were also recorded, as were optional comments by the observers.

Table 3. – Structure of the observation sheet
("teacher" refers to the teaching PSTs here)

Time	Phase / Phase transition	Description (of teacher's formulation of a goal, teachers' action or prompt) and object	Code (G, A, P)	Comment (e.g., deviating actual actions of the pupils)

3.1.3 Additional data sources

In addition to the observations from the staging, we use PSTs' presentations, the teaching materials they developed (in particular the worksheets), and their term papers.

3.2 Data evaluation

We conducted the data analysis in six steps, which we explain in this section:

Step 0	►	**Step 1**	►	**Step 2**	►	**Step 3**	►	**Step 4**	►	**Step 5**
Marked situations		*AT reconstruction*		*Patterns*		*Phenomena*		*PSTs' planning and staging*		*PSTs' discontinuity experiences*

Step 0: Pre-selecting staging situations that are marked as proving-related. Since pre-selecting situations as proving-related is a first step, in which analysis takes place, we are aware that there may exist instances where our understanding as researchers differs from the PSTs' understanding. We considered a situation as "marked by the PST", if at least one of the following three criteria applied:

(M1) The PST initiated "argumentative talk" (Hershkowitz et al., 2017, p. 25), for example when the PST weighs up different options against each other, (M2) the PST explicitly expressed (orally when giving instructions, in written task formulation, or in planning documents) that proving-related activities such as observing connections, conjecturing, reasoning, justifying, or proving are intended,

(M3) the PST collected or recorded results in class and explicitly expressed the goal that an object of a proving-related activity (e.g., an explanation, a proof, a conjecture) is supposed to be formulated collectively by the group of learners (e.g., based on the results of a previous work phase) or will be presented by the PST.

Step 1: Reconstruction. For the situations and sequences that we identified in Step 0, we reconstructed potential proving-related activities in the sense of AT, based on the observation sheets, the staging materials, and the planning and reflection documents of the PSTs. That is, we attempted to reconstruct a motive and a target object of an activity that could model the situation, and then analyzed the content-related actions in the teaching situation in terms of their possible contributions to the reconstructed target object. AT functions here as a reconstructive tool which, on the basis of the conceptual definitions and relations used in it with respect to "activity", "motive", "object" and "action goal", allows teaching situations and sequences to be evaluated in terms of whether such a reconstruction is possible at all, and if so,

- whether the reconstructed target object and motive can reasonably be considered as those of a proving-related activity in the sense of Sect. 2.3,
- whether the actions contribute recognizably (for the researchers) to producing the reconstructed target object in the course of the staging,

In line with our first research question (RQ1), we then selected those situations and sequences where according to these criteria a proving-related activity was not reconstructable. We give illustrations in the exemplary descriptions and analyses in Sect. 4.1.3.

Step 2: Identifying patterns through constant comparison. In the first pass, we looked at the selected situations and sequences at the level of behavioral description, that is, with regard to how PSTs acted with respect to the mathematical content. We obtained codes for behavioral patterns using the variant of the constant comparative method (Strauss and Corbin, 1998) employed by Lockwood et al. (2016). The data were independently precoded by two researchers, and the phenomena, codes, and codings were then agreed upon. (Note that the term "patterns" refers here to repetition *within our data set*. This does not imply that the participating PSTs have behavioral patterns *individually*.)

Step 3: Grouping patterns under overarching phenomena. Based on the AT reconstructions of the selected situations and sequences conducted in Step 1, we grouped the patterns under overarching phenomena in the sense that we conceive the patterns as instantiations of the phenomena.

In the remaining steps, the analysis concerns RQ2. Here we use AT as a background theory for providing possible explanations.

Step 4: From phenomena to PSTs in their role as planning and staging individuals. We use AT to propose for each phenomenon underlying circumstances on part of the PSTs who do the planning and staging work. That is, we interpret the teaching behavior described by the phenomenon as an expression of such circumstances (e.g., PSTs did not project (in the sense of Sect. 2.2) a suitable target object (in the sense of Sect. 2.3) for a proving-related activity onto the concrete staging situation).

Step 5: From PSTs als planning and staging individuals to discontinuities they experience between practices. In this step, we use abductive inference (see Magnani, 2001) to obtain explanations for the circumstances on the part of the individuum that underlie each phenomenon: The circumstances ("results" in abduction terminology) are explained as resulting from particular conditions ("cases" in abduction terminology) by a general "rule". Abductive inference, as the first step within the process of finding and testing explanatory hypotheses, is not inference in a logical sense. It is considered sufficiently reliable if – based on background knowledge – the proposed rule can be supported as valid and relevant and if it is plausible that the conditions are fulfilled (i.e., the cases occur). As such a "rule", we use an explanatory principle, which is supported by the chosen background theory (AT) and which we adapt to the topic of discontinuity experiences and their effects on teaching behavior. As possible conditions (cases), we identify certain discontinuities that PSTs experience between the mathematical practices they encounter. Explanations obtained through abduction need to be inductively tested in further research steps. We address follow-up research questions along these lines in the outlook section.

4 Results

4.1 Situations marked as proving-related in PSTs' lesson stagings (RQ1)

4.1.1 Phenomena and patterns

In our analysis of the six stagings A1, …, A6 from summer semester 2021 we were able to identify six patterns of PSTs' teaching behavior. We grouped the patterns into three phenomena. In this section we report these six patterns (see Table 4 for an overview) and in Sects. 4.1.3 and 4.1.4 we will present examples for selected patterns and develop AT analyses for the examples and for the respective patterns as variants of instantiations of the respective phenomenon.[1]

Table 4. Phenomena and patterns found in PSTs' stagings A1, …, A6.

Phenomena	Patterns	Sessions					
		A1	A2	A3	A4	A5	A6
Phenomenon 1: Missing out on opportunities for proving-related activities	(1.1) Receiving results and moving on	X	X	X	X	X	X
	(1.2) Leaving questions from students behind		X			X	X
	(1.3) Leaving opportunities unused in task construction	X		X	X		X
Phenomenon 2: Missing focus on content and on the conceptual core	(2.1) Strong emphasis on the methodological side of teaching	X		X		X	
	(2.2) Missing the conceptual core		X	X	X		X
Phenomenon 3: Functional structuring of proving-related activities is not exemplified	(3.1) Teaching by preparing written tasks or task sequences, whose dramaturgy does not initiate a proving-related activity	X	X	X	X	X	X

Phenomenon 1: Missing out on opportunities for proving-related activities

This phenomenon comprises three patterns, which occur both at the situational-spontaneous action level and at the level of reflexive planning action.

Pattern 1.1: Receiving results and moving on. Pre-service teachers receive and rate learners' answers, but they do not question the answers further or confront them with each other. Thus answers and results are not used as opportunities for proving-related activities.

Pattern 1.2: Leaving questions from students behind. Unexpected questions from learners are acknowledged as an element of classroom interaction, but they are left behind as opportunities for proving-related activities.

1. From now on the term "teacher"/"teachers" always refers to the PST/PSTs who do the teaching experiment. By contrast, the authors of this paper (in their role as teachers/instructors of the seminar) are referred to as "researchers".

Pattern 1.3: Leaving opportunities unused in task construction. Already in task construction opportunities for proving-related activities are not exploited.

To put the content of the patterns in perspective, let us emphasize that we are not assuming that every opportunity for a proving-related activity must necessarily be used in every lesson. However, in our analysis we consider situations that were marked as proving-related by the PSTs. The patterns thus describe situations in which there is a disconnect between intention and execution.

Phenomenon 2: Missing focus on content and on the conceptual core

This phenomenon comprises two patterns:

Pattern 2.1: Strong emphasis on the methodological side of teaching. The staging is methodically (and sometimes technically) overloaded with actions that are not related to the mathematical learning content..

Pattern 2.2: Missing the conceptual core. The staging contains mathematical actions related to the mathematical learning content, but these do not reach its mathematical core (see Sect. 2.3). In particular, the actions conducted or initiated by the teacher appear not to be conceived from an explanatory warrant with a view to foster learners' deeper mathematical understanding.

Phenomenon 3: Functional structuring of proving-related activities is not exemplified

We observed one stable pattern which occurred in all observed lessons. In its formulation, we use the term "dramaturgy" to refer to the way in which task sequences in a lesson are organized towards a goal.

Pattern 3.1: Teaching by preparing written tasks or task sequences, whose dramaturgy does not initiate a proving-related activity. The teacher prepares pre-formulated work assignments for individual or group work and then largely fades into the background during the staging. Neither the work assignments nor the classroom discussion of the results guide and clarify the interplay and contribution of single tasks and actions towards a target object.

4.1.3 Examples of patterns and AT reconstructions

As illustrations for the AT reconstructions of staging situations and sequences, we present and reconstruct in this section examples of the most common patterns in each of the three phenomena, that is, Patterns 1.1, 2.2, and 3.1.

(1.1) Receiving results and moving on

A1	A2	A3	A4	A5	A6
X	X	X	X	X	X

Learning content and marking: In Session A6 (scalar product) the learning content consisted of four basic mental models ("Grundvorstellungen") for the scalar product (Frohn, 2020), addressed in four learning stations S1–S4 that had been designed by the PSTs to take up and develop these models: projection (S1), orthogonality (S2), product (S3), and angle (S4). The session started with the coordinate form of the scalar product as its definition. The planning document marks a number of situations as proving-related (marking type M2): Regarding S4 it states the goal that "Pupils will explain the relationship between the scalar product and the cosine" (p. 13). For all stations, it sets the goal that "pupils will explain the effect of the scalar product" (p. 13) after pointing out the danger of going to a "procedural conception too quickly" (p. 10). Task assignments contain markings such as "Find a possible explanation" (in the task sheet for S4, marking type M2). In plenary, when the results of S2 are collected, the PST asks "Why?" when a pair of vectors had been empirically recognized as being orthogonal (marking type M3).

Execution: In the plenaries after the learning stations, the results were reviewed without further questions on how students arrived at these results. For example, in plenary for Station 4 (at minute 37 of the lesson, excerpt of the observation sheet of Researcher 1):

> PST: "Results!"
>
> Student 1: (reports his results and states a conjecture concerning the sign of the scalar product)
>
> PST: "This is correct. Results!"
>
> Student 2: (reports his conjecture that pairs of vectors have the same scalar product if they enclose the same angle)
>
> PST: "OK."

No explanations were requested and there was no discussion of what might be behind the conjectured relationships. Also, the "Why" question regarding S2 was left unanswered.

AT reconstruction: Learners' development of basic mental models for mathematical concepts can be seen as a necessary prerequisite for meaningful reasoning and proving, and in this sense, it can be conceived of as a motive of a proving-related activity. As the session started from the algebraic definition of the scalar product (in coordinate form), the target object of such an activity would need to relate this algebraic definition to images associated with the mental models (answering, for example, why a scalar product of zero is related to the observed orthogonality of vectors). From the lesson staging (including task construction and PST's actions in the lesson), however, an appropriate target object in this sense cannot be reconstructed, as the mere results do not establish a connection of the algebraic definition with the basic mental models.

(2.2) Missing the conceptual core

A1	A2	A3	A4	A5	A6
	X	X	X		X

Learning content and marking: In Session A3 (half-life), the learning content was the half-life in the context of exponential functions, which was concretized in two experiments (entitled "beer foam" and "dice-throwing"). In group work, students gathered experimental data and subsequently used it for calculations such as "Calculate after how many seconds the height of the beer foam is 1.75 cm". The planning document marks part of the experiments as proving-related, when it sets the goal that "the pupils formulate a hypothesis about the half-life. Each group agrees on a hypothesis that is to be tested below" (M2). Correspondingly, the task sheets mark tasks as proving-related, when they require to "Formulate a conjecture as to after how many rounds there will be 75 cubes left" (see Fig. 2, M2).

Execution: In the main part of the lesson, the students worked in two groups, each of which was assigned one of the experiments. The work in the groups was guided by task sheets. Fig. 2 shows the task sheet for the "dice-throwing" group. The experiment (31 dice rolls) was conducted using a web application.

Worksheet – Rolling the dice

(1) Make a conjecture as to after how many rounds there will be 75 cubes left.
(2) Complete the table.

Round	Number of dice with the number six per round	Number of remaining dice
0	0	150
1		
2		

[…]

30		
31		

(3) Graph the measured values (number of remaining cubes as a function of the number of rounds) and plot the 1st half-life.
(4) Set up a decay function. Use the first half-life from the table.

Figure 2. – Task sheet for the "dice-throwing" working group (translated from German, the table in Task (2) contains 31 rows in the original) .

AT reconstruction: In this session, the motive to model everyday phenomena mathematically through decay functions can be reconstructed from both planning documents and the PSTs' input. Such modeling would necessarily include a discussion of why the phenomena under consideration exhibit exponential behavior in the first place. For example, one may wonder how dice throwing relates to half-life – and in fact it is from our point of view part of the conceptual core of the matter to establish this relationship, in this case by reasoning why the rules of the used dice-throwing game are the decisive reason for exponential decay. Such reasoning, however, was not addressed, neither in group work or discussion (each of the groups was observed by one of the two researchers) nor in the subsequent plenary. In the task sheet (Fig. 3) exponential decay seems to be implicitly assumed (Task 3 mentions "half-life" and Task 4 asks for a "decay function"). We are aware that we are making a normative determination when we declare a specific content aspect to be the conceptual core. In the present case, we base this determination on the fact that without a justification of the exponential character of the experimental processes, the measurements and calculations of this lesson could also be carried out with other functions in the same way, thus rendering the experiment pointless.

(3.1) Teaching by preparing written tasks or task sequences, whose dramaturgy does not initiate a proving-related activity

A1	A2	A3	A4	A5	A6
X	X	X	X	X	X

Learning content and marking: In session A2 (odd power functions) the learning content consisted in properties of power functions with odd exponent (i.e., functions of type x^{2n-1}). According to the planning document, the main part of the lesson takes place in individual and group work (29 minutes out of a total of 39 minutes, using the jigsaw method). These parts are marked as proving-related by setting the goal that "Pupils independently develop and present findings on various power functions with odd positive exponents" (p. 22, M2) and by explicitly relating the goal of "Identifying commonalities and transferring them to the overall concept" to the basic mathematical experience ("Grunderfahrung") of getting to know mathematics as a "deductively ordered world" (p. 24, referring to Winter, 1995).

Execution: Individual and group work is based on tasks sheets. Fig. 3 shows the sheet for expert group A. The task sheets for groups B and C differ from A only in the values of the table in Task 2; the tables for groups A, B and C describe the functions x^3, x^5, and x^7 respectively.

AT reconstruction: The goals stated in the planning documents are consistent with the motive "work out properties of odd power functions" for an activity whose target object is a characterization of these functions by their properties. The planning documents do not clearly indicate whether the properties are to be developed as conjectures or with justification. This is not a crucial limitation for our analysis, because proving-related activities are conceivable for both possibilities. In either case, the prepared task sequence raises questions, when viewed towards the target object: On what basis can a conjecture be obtained in Task 1? How would students investigate the condition that the exponent be odd, if in Task 3 properties of the function are read from a table? Moreover, both researchers' notes in their observation sheets show that at no point during the entire lesson, the empirically found properties were related to the condition that the exponent be odd. It is therefore unclear how the provided task sequence could lead to a sequence of actions and goals that contribute to producing the target object (see Fig. 4, compared with Fig. 1 in Sect. 2.2).

Power functions with odd positive exponent *Expert A*

$f(x) = x^{2n-1}$

(1) Make a conjecture: What influence does the odd positive exponent have on the function graph compared to the known quadratic function?
(2) Which function is described by the following table of values? Fill in the missing values.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
y	-8		-1		0		1		

(3) Read the following properties from the table of values without drawing the graph: domain, range, zeroes, monotonicity, symmetry
(4) Draw the graph of the function in the given coordinate system.

Figure 3. – Task sheet for expert group A from Session A2 (translated from German).

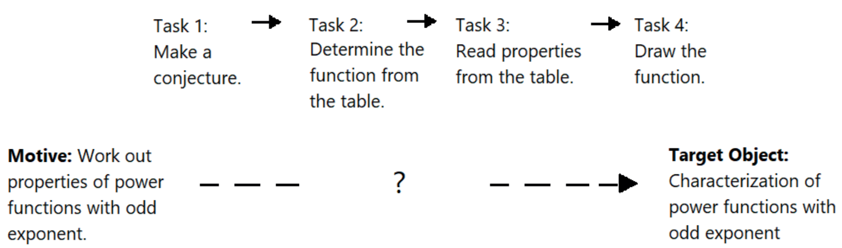

Figure 4. – Dramaturgy of tasks in A2 that does not lead to a sequence of actions and goals that contribute to producing the target object.

4.2 Activity-theory based interpretations and proposed explanations for the phenomena as effects of discontinuity experiences (RQ2)

In the following, we elaborate proposals for explaining the three reported phenomena that point to discontinuity experiences of PSTs as possible causes. The proposed explanations are to be seen as a first step in providing appropriate explanatory hypotheses. At this point, we do not aim at making any site-specific or even individual, strong causal statements. Further steps of inductive testing will need additional data of a different kind (see Sect. 5.4).

Our two step explanatory approach employs in the first step (Sect. 4.2.1) AT interpretations of the three phenomena that are grounded in the AT reconstruction of staging situations and sequences subsumed under their respective patterns, and in the second step (Sect. 4.2.3) a more general AT explanatory principle (stated in Sect. 4.2.2).

4.2.1 AT interpretation of the phenomena and exemplary patterns

We present in this subsection AT interpretations of the three phenomena that we conducted on the basis of our AT reconstructions of the staging situations and sequences. We interpret the identified disconnects between PSTs' intentions (expressed by marking as proving-related) and their execution (where constitutive elements of a proving-related activity from an AT-perspective were not reconstructible) as indicative of difficulties that PSTs have in assessing and mediating between motives, target objects and action goals.

Phenomenon 1: Missing out on opportunities for proving-related activities

Based on our AT reconstructions like the one presented for the case of A6, Patterns 1.1-1.3 can be understood as variants of a teaching behavior described by Phenomenon 1: The teacher is missing out on opportunities for proving-related activities. The AT perspective provides an interpretation of phenomenon 1 with a focus on *target objects:* We interpret this teaching behavior as an expression of the circumstance that the teacher does not project a suitable target object of a proving-related activity onto the teaching situation (e.g., a situation where learners present results of prior working phases, as in Pattern 1.1). They do not anchor their planning in a target object, and as a consequence, they do not utilize the potential of tasks or they do not take up results of partial actions or unplanned argumentative contributions from learners (such as conceptual questions or why-questions) in the further course of the staging and bring them together to form such a target object.

Phenomenon 2: Missing focus on content and on the conceptual core

Based on our AT reconstructions like the one presented for the case of A3, Patterns 2.1 and 2.2 can be understood as variants of a teaching behavior described by Phenomenon 2: a missing focus on content and on the conceptual core. That is, the actions conducted or initiated by the teacher, as well as the target object (e.g., a convincing argument for a conjecture or a justified mathematization of a given real world situation) initially match those of a proving-related activity, but action goals and the handling of the action outcomes lack a clear focus on the conceptual core of the learning content. The perspective of AT provides an interpretation of Phenomenon 2 with a focus on *motives:* We interpret this teaching behavior as an expression of the circumstance that the teacher's focus is on the implementation of motives that are only to a small extent content-related, such as "constructing knowledge actively". In the end, the motive implemented by the teacher appears to be rather detached from a proving-related activity: It seems to be less determined by a need to understand corresponding core aspects of the specific learning content, but more by a general pedagogical need to actively construct knowledge.

Phenomenon 3: Functional structuring of proving-related activities is not exemplified

Based on our AT reconstructions like the one presented for the case of A2, Pattern 3.1 can be understood as variants of a teaching behavior described by Phenomenon 3: The functional structuring of proving-related activities is not exemplified by the chosen sequences of actions, that is, even though a target object as well as a corresponding motive might have been considered by the PST during planning and might even have been communicated to the learners, the ways, means and goals of action are not chosen, combined and executed in such a way that they effectively contribute to producing the target object. The perspective of AT suggests an interpretation of Phenomenon 3 with a focus on *constructions of meaning:* We interpret this teaching behavior as an expression of the circumstance that the teacher does not draw on individual constructions of meaning in order to create a dramaturgy of actions and tasks that contributes to a target object of a proving-related activity.

4.2.2 Base and structure of the proposed explanations

We now abductively infer what kind of discontinuity experiences PSTs might have made that could have brought about the circumstances underlying the observed phenomena. Our abductive explanations for the three phenomena have a common structure, expressed in terms of "result", "case", "rule" (see Sect. 3.2, Step 5). We start from the obtained result:

- *Result:* In the sense of our AT interpretation of Phenomena 1–3, PSTs have difficulties in recognizing, selecting or creating motives, target objects, and a dramaturgy of tasks and partial actions of proving-related activities for lesson staging.

Abduction explains the result by the hypothesis that it is caused by a presumed case as a consequence of a general rule. The "rule" we use is an explanatory principle based upon the insights from AT (see Sect. 2.2).

- *Case:* PSTs have made specific discontinuity experiences with regard to objects, motives, or ways and means of action between proving-related activities at school and university.
- *Principle ("rule"):* In order for PSTs to be able to successfully stage and initiate proving-related activities in their later teaching, the following is required: First,

> PSTs need to experience and perform proving-related *actions* in school and university as embedded in proving-related *activities*. They need to develop suitable constructions of meaning, which they can access in their later role as teachers when planning to stage such activities. Second, PSTs need opportunities to reflect on these activities in terms of their constitutive elements, so that they can establish connections between school and university practices.

After the abductions, we explain for each phenomenon why it is plausible that the respective case in fact occurs.

4.2.3 Proposed explanations for Phenomena 1–3

Phenomenon 1

We interpreted Phenomenon 1 in the sense that the teacher does not project a suitable object of a proving-related activity onto the staging situation (*result*). Therefore, discontinuity experiences that are related to target objects of proving-related activities particularly come into question as possible causes for such circumstances on the part of the PSTs. We can abductively conclude the following possible *case*: PSTs experience an existing or perceived discontinuity regarding target objects of proving-related activities between practices they have experienced in school and university.

It is plausible that this case actually occurs: At university, PSTs may have experienced a consistently growing and globally organized corpus of deductively justifiable knowledge based on well-grounded axioms as an overarching ideal target object of proving-related activity, to which the target objects and goals of specific proving-related activities and actions make a recognizable contribution. At school, on the other hand, they will have experienced proofs as deductive derivations usually only in an isolated way in the context of special lessons, for example on the Pythagorean theorem group, where the proving-related activities are directed to verifying the general validity of the mathematical statements in question (see, e.g., Knuth, 2002; Grundey, 2015, p. 129). PSTs are likely to have experienced the demand to aim for informal proofs at school, as the notion of "sound informal proofs" is a well-developed concept in (West) German mathematics educational discourse ("inhaltlich-anschauliche Beweise", Wittmann, 2021). PSTs, however, may not have seen many concrete realizations of this concept, depending on their school and university experiences.

Phenomenon 2

We interpreted Phenomenon 2 in the sense that the teacher's focus is on the implementation of motives that arise from general pedagogical needs, for example the need to actively construct knowledge, rather than from a need to understand core aspects of the specific learning content (*result*). Here, we can abductively conclude the following possible *case*: PSTs experience an existing or perceived discontinuity regarding motives of proving-related activities between practices they have experienced in school and university.

It is plausible that this case actually occurs: At university, they might experience the over-arching motive of embedding mathematical concepts and propositions into in principle globally deductively orderable mathematical theories, where gaining certainty and under-standing about general validity and theory building are complementing facets. By contrast, during their school internships they might experience an orientation towards broad general educational goals in German school standards and curricula, which they may perceive as a main legitimation and planning base for classroom staging. They are confronted with rather unspecified quality dimensions for teaching like "learners' activation" (see Renkl & Atkinson, 2007) and with a tradition of planning from competences and methods rather than from content, which might place different motives and goals for the staging of proving-related activities in the foreground (e.g., learners' independent action or reference to everyday life).

Phenomenon 3

We interpreted Phenomenon 3 in the sense that the teacher does not draw on individual constructions of meaning in order to create a dramaturgy of actions and tasks that con-tributes to a target object of a proving-related activity (*result*). Thus, we can abductively conclude the following possible *case*: PSTs experience an existing or perceived disconti-nuity regarding (the interplay of different) ways and means of action of proving-related activities between practices they have experienced in school and university.

It is plausible that this case actually occurs: At university, they experience mainly formal-de-ductive arguments as ways and means of action of proving-related activities as it is reflect-ed in rather product-oriented undergraduate mathematics lectures and exercises, which despite various recent developments (see Wasserman et al., 2023) is still the dominant focus in undergraduate university mathematics education (Reiss & Törner, 2007, p. 436; see also Gildehaus et al., 2021, Pritchard, 2010). Thus, PSTs might only notice a limited type and amount of ways and means of action, largely excluding pre-formal or non-deductive proving-related actions that constitute proving as a complex process of generating under-standing and which could be perceived as inspiring also for school teaching: "However, exposure to mathematical practices does not occur automatically by virtue of a course deal-ing with university level content. These practices are often left implicit" (Wasserman et al. 2023, p. 726). At school, by contrast, they rarely experience formal deductive arguments in received or observed mathematics lessons. When they experience pre-formal, informal, or experimental ways and means of action of proving-related activities at school, these will rarely occur as steps in a process that leads to a more formal or deductive argument.

5 Discussion

5.1 Summary of findings

Our first research question concerns situations occurring in PSTs' lesson stagings that, even though marked by the PSTs as proving-related, cannot be reconstructed as proving-related activities. Our findings provide a phenomenology comprising three phenomena in teaching behavior which manifest themselves in different patterns. Some of the patterns directly correspond to specific types of staging situations (e.g., comparing results from task work, working on prepared working sheets, employing methods, receiving unexpected questions from learners). Based on an AT reconstruction of these situations and patterns, we provided AT analyses of the three phenomena in terms of motives, objects, ways and means of action and constructions of meaning, in order to relate the phenomena to differences between practices concerning proving-related activities. Our second research question asks for explanations that AT provides for these phenomena as possible effects of discontinuity experiences that PSTs might have made. Using abductive inferences we suggested specific discontinuity experiences as possible causes for the observed phenomena and patterns on the levels of motives, objects, and ways and means of action. These proposed explanations suggest that it is crucial to mediate PSTs' discontinuity experiences that may occur when proving-related activities in school and university were driven by different motives and directed towards different objects. Also, concrete and explicitly reflected action experience with proving-related activities at university seems important for students to develop constructions of meaning for such activities.

Our results confirm findings known from the literature (Ko, 2010; Bieda, 2010; Buchbinder & McCrone, 2020) that teachers have difficulty implementing proving-related activities in their classrooms. Our paper makes two specific contributions that go beyond what is known so far: First, rather than looking at stagings that are specifically declared as "proving lessons", we deliberately examined in our study how participants behaved when planning and implementing lessons on a self-selected topic. This seems relevant to us because proving-related activities should not occur only in isolated places in mathematics instruction. As a second contribution, our findings describe and analyze concrete phenomena and patterns of behavior that occurred among the PSTs we observed. On the one hand, this opens up the possibility of focusing more specifically on these patterns in further studies. On the other hand, it shows us concrete aspects regarding mediating elements that we can address in course design; the activity-theoretical explanations provide information about the way in which motives, objects, and constructions of meaning should be dealt with in specific courses integrating disciplinary, school-content-related, subject didactic, general educational and practical planning perspectives to build a hinge for mitigating discontinuity and founding coherence experiences of PSTs.

Let us note again that the findings from the observed lesson stagings may initially sound negative and deficit-oriented. It is true that the research questions ask for problematic aspects of the lessons, since our focus is on uncovering them and on drawing conclusions for course designs in PST training. However, our findings do not say that the observed lessons were unsuccessful overall, as many of them were successful with respect to aspects other than those focused on here (for example, with regard to collaboration in groups when working on tasks).

5.2 Limitations and strengths of the study

The data we used in this study are suitable for obtaining a phenomenology that can serve as a basis for an abductive inference of possible explanations for the phenomena in terms of potential discontinuity experiences. Based on our data, we cannot determine which discontinuity experiences PSTs have actually had individually and thus make causal statements about individual PSTs. However, this limitation is also related to a strength of the AT framework: If we consider it plausible that similar discontinuity experiences also occur in other PSTs, then our AT reconstructions and explanatory principle enable us to draw informative and plausible conclusions that are relevant for university teacher education and on possibilities of addressing them (as the first step in a process of finding and testing explanatory hypotheses). At this point, we do not rely on evidence for individual discontinuity experiences, which may be difficult to access empirically.

As another limitation, the capstone course in which we conducted the observation is not an authentic classroom situation in that we do not observe students in their autonomous teaching at a school, but in a university course. So in particular the addressees of the lessons are not pupils but fellow PSTs. Also, since the PSTs are performing the assignment as part of a university program, they are in a situation where their performance will be graded. Thus, on the one hand, the university context is a limitation of our study in terms of its data source. On the other hand, it is also a strength, because the participants were subject to far fewer constraints in their lesson planning than is the case with teachers in ongoing classes, where curricular requirements, a heterogeneous student body, possible time constraints, and much more have to be taken into account. In this sense, our sample may even be particularly suitable to draw inferences from patterns in their teaching behavior back to possible causal influence of discontinuity experiences, as lessons in an authentic school context are subject to more (and partly non-subject-related) constraints, which presumably require more compromises. If the patterns we found were to occur there, we would have to consider, for example, whether some of them resulted from the fact that the teachers had made decisions with regard to specific learners or from the fact that the curriculum had imposed certain requirements on them.

5.3 Implications for pre-service teacher education

Our framework has enabled us to uncover specific phenomena and patterns in PSTs' behavior in planning and staging mathematics lessons that can be interpreted as indications that the participating PSTs have not sufficiently developed a proof-oriented disposition. Also, it enabled us to elaborate possible explanations in terms of discontinuity experiences. The AT reconstructions as well as the proposed interpretations and explanations reveal starting points for developing elements in university teacher education that might mitigate effects of discontinuity experiences by explicitly addressing the three levels of motives, objects, and ways and means of proving-related activity.

We argue that our findings underline the importance of having PSTs develop specific content-related knowledge for their later teaching, in particular sufficiently deep and interconnected mathematical knowledge and understanding of mathematical concepts and ways and means of action of proving-related activities, as a basis for identifying and using motives, target objects and ways and means of action (preformal, heuristic, generic) and for developing constructions of meaning for proving-related activities in the classroom. University courses for PSTs should include input and reflection on concrete examples for such target objects and ways and means of action (see Schwartz et al., 2008, p. 186) for various content areas and grades. They should promote PSTs' concrete experience of gaining personal understanding from proving-related activities, as a complementing motive to the general motive of theory building, thus balancing possible perceptions of PSTs that understanding and theory building appear as competing instead of complementing motives in school mathematics practices they encounter. Comparative results, for example from Schwartz et al. (2008) actually indicate a tendency towards such perceptions (see Schwartz et al. 2008, p. 807f.). To these ends, university teacher education needs to overcome a complete separation of mathematics and mathematics didactic courses (further arguments for the latter can also be found in Wasserman et al., 2023, p. 721). Course designs such as those presented by Buchbinder and McCrone (2020, 2023) and Bauer and Müller-Hill (2024) address this issue with respect to proving-related activities. With regard to the goal of building connections between school mathematics and university mathematics for students, there have been a number of approaches in recent years that use interface tasks as bridging elements (Ableitinger et al., 2013; Bauer, 2013a, 2013b; Álvarez et al., 2020; Hoffmann & Biehler, 2023).

In addition to that, we argue that our findings also indicate the role of the quality of experiences in internship phases or school-practical exercises (see, e.g., Doll et al., 2020; Gröschner & Schmitt, 2010) and underline the need to interlink university studies and school internships more strongly and directly to this end. The way and extent to which such interlinking can actually be realized at different universities varies. However, it appears to be crucial that during their internship phases, PSTs are explicitly guided in conducting classroom observations on proving-related activities employing subject-specific and content-related foci (and not only with a view to general issues, like, e.g., classroom management). Also, they should be guided in reflecting critically on possible experiences from stagings of proving-related activities, for example when the emphasis of the staging is more on fitting actions into the methodological paradigm of questioning-developing instruction than on directing actions to target objects of such activities in a recognizable way (see Heinze & Reiss, 2004, for results that indicate tendencies for such an emphasis in German mathematics classrooms). In a comparative study, Blömeke et al. (2008, pp. 755 and 757-759) find that PSTs in Germany put a salient focus on aspects of methodological and instructional organization as well as on the (presumably standard-compliant) formulation of learning objectives when evaluating lesson plans, whereas aspects such as mathematical content and conceptual understanding are used as criteria only to a relatively small extent. We argue therefore that PSTs should explicitly practice how they can specify and employ general education goals and general quality dimensions with respect to concrete mathematical content when they plan proving-related activities.

The AT-specific perspective on mediating elements that emerges from our findings emphasizes the need to think holistically about mediating elements that provide the following key components: authentic individual action experience with proving-related activities in school and university, overarching reflection on these experiences using the AT concepts of motive, target object, need, goal, ways and means of action, and suitable process-models for proving-related activities, as well as concrete transfer into staging. We understand this as a way to address the question recently posed by Wasserman et al. (2023, p. 726): "The transfer of one's own experiences [of mathematical practices at the university level] (including in more advanced mathematics courses) into teaching expertise is an open question." As a concrete contribution in this sense, we propose in Bauer and Müller-Hill (2022b, 2024) detailed design principles, specific course designs, and empirical results from seminar courses. These courses aim to engage PSTs in authentic processes of developing and understanding mathematical conjectures, arguments and proofs at school and university levels, to guide them in comparative reflection of such proving-related activities with a view towards motives, target objects, and the interplay of goals of action, and to foster PSTs' ability to relate their respective experiences to their own motives for learning mathematics as well as to educational goals and motives for the mathematics classroom. Such engagement aims at providing PSTs access to concrete as well as to overarching motives and target objects of individually meaningful proving-related activities in school and university mathematics practice as two integrable domains of experience. It is meant to support PSTs in developing constructions of meaning for proving-related activities that are helpful in mediating between these domains as a basis for staging proving-related activities in the mathematics classroom.

5.4 Outlook

As already stated, the proposed explanations from Sect. 4.2 need to be inductively tested in further research to support site-specific or even individual, stronger causal statements concerning the actual existence and relevance of the abductively inferred discontinuities. Direct inductive testing may turn out to be difficult: It might in principle be hard to reliably trace back individual discontinuity experiences, the study biographies of PSTs can be rather heterogeneous even for PSTs at the same university, and the influence of experienced courses on perceived images and discontinuities of and between practices appears to be highly individual (as at least indicated, e.g., by studies like Buchholtz, 2017). One possible approach to indirect inductive testing concerns further investigation of the concrete, AT-informed proposals for mediating elements in PST course design like those developed in Bauer and Müller-Hill (2022b, 2024). Here, it would be very desirable to have long-term data that could show how PSTs who experienced the implemented mediating elements in a holistic framework design actually bring these experiences to bear in their later work as teachers. One promising approach could be case studies like in Buchbinder and McCrone (2020), which could also complement our data to investigate connections between concrete individual discontinuity experiences and teaching behavior in single cases.

Our results suggest further research questions concerning the staging of proving-related activities in actual school practice. It appears that proving-related activity has been much less empirically investigated than dedicated "proof lessons". Research studies focusing on the following questions would help to fill the gap: On which occasions and in what ways do in-service teachers enact proving-related activity in their teaching? What image do in-service teachers actually have of potential target objects of proving-related activities for school mathematics teaching, and how are these embedded in an overall picture of the (kind of) mathematical knowledge and practices that they want their pupils to develop in the classroom?

Author note

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