Scriptwriting as a Catalyst for Linking Undergraduate and School Mathematics

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Scripting tasks are a powerful tool for both mathematics education researchers and teacher educators, in part because the resultant dialogues provide insight into the scriptwriters' mathematical understanding and pedagogical inclinations. In this paper, we argue that scripting tasks used in mathematics education also provide an opportunity to deliver follow-up lessons that link undergraduate and school mathematics. These lessons facilitate mathematical connections by building directly on scriptwriters' experiences, as captured in their dialogues, and in turn, enrich their teaching practice.

Keywords: scripting tasks, mathematical knowledge for teaching, mathematics teacher educators

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La escritura de guiones como propulsor para vincular la matemática pregrado y escolar

Las tareas de escribir un guión son una herramienta poderosa tanto para los investigadores en educación matemática como para los formadores de docentes, en parte porque las conversaciones derivadas brindan información sobre la comprensión matemática y las inclinaciones pedagógicas de los guionistas. En este artículo, sostenemos que las tareas de escribir un guión utilizadas en la educación matemática también brindan una oportunidad para impartir lecciones de seguimiento que vinculan las matemáticas de nivel universitario y escolares. Estas lecciones facilitan las conexiones matemáticas al aprovechar directamente las experiencias de los guionistas, tal como se capturan en sus diálogos, y, a su vez, enriquecen su práctica docente.

Palabras-claves: tareas de guión, conocimientos matemáticos para la enseñanza, formadores de docentes de matemáticas

La création de scénario comme catalyseur pour lier les mathématiques du premier cycle et les mathématiques scolaires

La créations de scénario est un outil puissant à la fois pour les chercheurs en éducation mathématique et pour les formateurs d'enseignants, en partie parce que les dialogues qui en résultent donnent un aperçu de la compréhension mathématique et des inclinations pédagogiques des scénaristes. Dans cet article, nous soutenons que l'utilisation de scénarios dans l'enseignement des mathématiques offre également la possibilité de proposer des leçons de suivi reliant les mathématiques du premier cycle et celles de l'école secondaire. Ces leçons facilitent les liens mathématiques en s'appuyant directement sur les expériences des scénaristes, telles qu'elles sont capturées dans leurs dialogues, et enrichissent ainsi leur pratique pédagogique.

Mots-Clés : tâches de script connaissances mathématiques pour l'enseignement; formateurs d'enseignants de mathématiques

Introduction

The potential of scripted dialogues for teaching and learning mathematics was realized as early as the fourth century BC, when Plato recreated Socrates conversation with his students. Modern authors—see, for example, Lakatos' *Proofs and Refutations* (1976) and Pólya's *How to Solve it* (1945)—have continued to leverage dialogue as a didactical tool at the intersection of mathematics and pedagogy. More recently, contemporary mathematics education researchers have used dialogue with supporting commentary to foster discourse on learning and teaching (e.g., Zazkis & Koichu, 2015; Mason, 2018).

In the Socratic tradition, these exemplary dialogues tend to describe an interaction between a knowledgeable expert and a pupil. The expert characters often represent the viewpoint of the scriptwriters, and thus, their explanations are reflective of a comprehensive understanding of the topic at hand. Recent applications of scriptwriting to mathematics education research, however, have taken to analyzing dialogues written by student authors; in this way, the dialogues act as a tool for gaining insight into the scriptwriters' developing ways of thinking about mathematics and pedagogy.

The emergence of these student-written scripts began with the *lesson play* (Zazkis et al., 2009; Zazkis et al., 2013). Originally introduced as a more robust form of lesson planning, lesson plays required teachers to first anticipate where in their lesson they would need to facilitate an important discussion with their class or attend to a particular question or misconception of a student. Then, they scripted this interaction in the form of a dialogue. Lesson plays are a means of instructional planning that simultaneously serve as an approximation of practice (as in Grossman et al., 2009), unlike the traditional method of organizing a future lesson's planned activities in an outline.

Scripting tasks extend the use of student-written dialogues beyond lesson planning to more general mathematical settings. In a scripting task, scriptwriters are invited to continue a hypothetical mathematical dialogue that is already underway. The first lines of this existing dialogue are provided to the scriptwriters and introduce an unexpected question, observation, or disagreement about the mathematical setting. The scriptwriters then seek to bring the problematic situation to a satisfying resolution within the remaining dialogue.

Scripting tasks have an extensive and varied history as tools for mathematics education and the research thereof (e.g., Bergman et al., 2023; Brown, 2018; Buchbinder & Cook, 2018; Kontorovich & Zazkis, 2016). Zazkis and Marmur (2021) review the benefits of such tasks, as described in the literature, for three different populations of interest. For researchers, scripting tasks are a "rich data source that can be examined from various perspectives" (Zazkis & Marmur, 2021, p. 85)—that is, they not only provide researchers with insight into the scriptwriters' mathematical understanding but also into their perception of what constitutes effective pedagogy in practice. For students who compose the scripts, crafting a dialogue facilitates the growth of mathematical knowledge within an applicable pedagogical setting, and conversely, the development of pedagogical experience in a relevant mathematical context. We call attention to the fact that, for both populations, scripting tasks occupy a unique space at the intersection of mathematics and pedagogy.

The cross-disciplinary benefits of scripting tasks are especially evident for the third population considered by Zazkis and Murmur (2021): mathematics teacher educators. Instructors who are responsible for the training and development of teachers can, like mathematics education researchers, use scripting tasks to better understand students' mathematical and pedagogical content knowledge. Teacher educators, however, then have the unique opportunity to immediately leverage this insight to broaden and advance the emerging competencies of teachers.

We submit that scripting tasks can also be leveraged in mathematics teacher education in order to highlight connections between secondary and undergraduate mathematics. In such tasks, the teacher-character's dialogue within a prompt can point to a connection to more advanced mathematics, but this is not essential. Based on the completed scripts, a further connection to relevant mathematical ideas can be brought to teachers' attention in follow-up instruction, which may include explicit connections among mathematical topics or concepts as well as ideas on how these can be utilized pedagogically.

For example, Zazkis and Marmur (2018; 2021) describe the ways in which they use scripting tasks as a springboard for future lessons designed to make such connections. In the following section we summarize how they accomplished this goal with a scripting task based on secondary students' perception of the function concept. We then explore how additional connections to teaching might appear in further study of the function concept.

After considering how the work of Zazkis and Marmur (2018; 2021) could be extended, the next sections introduce and extend other studies that feature scripting tasks. In these studies, Zazkis and Cook (2018) use scripting tasks in a mathematics classroom to probe undergraduate students' understanding of zero divisors; meanwhile, Kercher et al. (2023) features scripting in a mathematics education classroom as a way to record teachers' investigations of star polygons. In each case, we consider how follow-up lessons could build on the thinking exhibited in the scripted dialogue in order to highlight connections to secondary mathematics.

On functions: connections from a scripting task

Zazkis and Marmur (2018; 2021) presented the prompt seen in Figure 1 to a group of secondary teachers in a professional development course. In addition to continuing the presented dialogue, the teachers were asked to explain the pedagogical choices made by the teacher-characters in their scripts and to elaborate on how their own understanding of the function concept might extend beyond what is evident from the dialogue.

Teacher:	Consider the following table of values. What function can this describe?
Alex:	y = 3x
Teacher:	And why do you say so?
Alex:	Because you see numbers on the right are 3 times numbers on the left
Jamie:	I agree with Alex, but is this the only way?
Teacher:	…

x	y
1	3
2	6
3	9
4	12
5	
6	

Figure 1. – The prompt used by Zazkis & Marmur (2021, p. 87).

Zazkis and Marmur (2018; 2021) considered the dialogues and supplementary explanations in order to identify two themes that characterized certain limitations of the scriptwriters' understanding of function. In response to each theme, Zazkis and Marmur (ibid.) planned and implemented a follow-up lesson that sought to address these limitations while simultaneously presenting the teachers with important mathematical connections.

Theme 1: Expanding Teachers' Conceptions of Function

Within the first theme, Zazkis and Marmur (2018; 2021) found evidence suggesting that the personal example spaces (see Sinclair et al., 2011) of the scriptwriters were mainly limited to functions continuous on the entire real line. Furthermore, certain interactions within the dialogue—such as when a teacher-character accepts a student's claim that, for any function, "Each output can only have 1 input" (Zazkis & Marmur, 2021., p. 89)—indicated that some teachers also had incomplete or incorrect understanding of the definition of function. Taken together, these observations resulted in dialogues that explored only a small subset of possible functions that could be represented by the chart in Figure 1.

The follow-up activity associated with this theme first tasked the teachers, working in groups, to write a formal definition of function. Then, the groups were presented with a collection of definitions sourced from both mathematical history and modern textbooks. These definitions were supplemented by incorrect "definitions" that exhibited some of the explicit errors observed in the scripted dialogues. The teachers then sorted the collection of definitions into categories and compared them to their own definitions. Ultimately, the goal of this activity was to refine the teachers' concept image and concept definition of function (in the sense of Tall & Vinner, 1981) to include a wider variety of functions and to more correctly align with the formal concept definition, respectively.

Achieving this goal highlighted an important connection between secondary and undergraduate mathematics. Several of the definitions presented to the teachers conceptualized functions as sets of ordered pairs. Such a definition is not likely to be appropriate for secondary mathematics classrooms; however, this advanced perspective helped draw attention to the existence of functions with discrete or disconnected domains. The historical definitions—in particular, the way they were seen to capture additional nuance over time with more precise language—emphasized the arbitrary nature of the relationship captured by a function. By responding to evidence from the dialogues that teachers held a view of function limited to continuous functions on the real numbers, Zazkis and Marmur (2018; 2021) grew teachers' mathematical content knowledge with an appropriate follow-up lesson.

Theme 2: Constructing a Polynomial

The second theme was related to many teachers' belief that only a linear function can contain the set of points described in Figure 1. However, the realization that the set of points itself already defines a function raised a question about the possibility of other nonlinear functions that contain the four colinear points given in the prompt. Zazkis and Marmur (2018; 2021) report on scripts wherein a teacher-character relied on a computer program to generate such a polynomial; purported to have found a higher-degree polynomial, which was revealed to be linear when simplified; and argued for the existence of such a polynomial without attempting to find a specific example. Zazkis and Marmur (ibid.) recognized in these responses an opportunity to broaden the scriptwriters mathematical understanding by introducing methods for constructing a family of nonlinear polynomials that all could contain points from the chart in Figure 1.

In the follow-up lesson that was designed to accomplish this goal, the teachers were initially asked to find a nonlinear polynomial $f(x)$ that had four distinct zeros. Then, they searched for a way to use f as the foundation for a function that passed through the four colinear points from the table in Figure 1. They eventually created a linear combination of f and the linear function $g(x) = 3x$ to yield the family of functions $h(x) = kf(x) + g(x)$. This process is visualized in Figure 2. Following this first activity, the class attended to an example, pulled directly from a script, of a third-degree polynomial that appeared to match the values given in the prompt. A discussion on the fundamental theorem of algebra (in particular, the fact that a third-degree polynomial has at most three real roots) explored how the teachers could use their knowledge of this advanced mathematical fact to recognize that the proposed function cannot include all the points in Figure 1. Thus, the conclusion could have been reached without conducting algebraic manipulations until the expression is recognized as a "linear function in disguise".

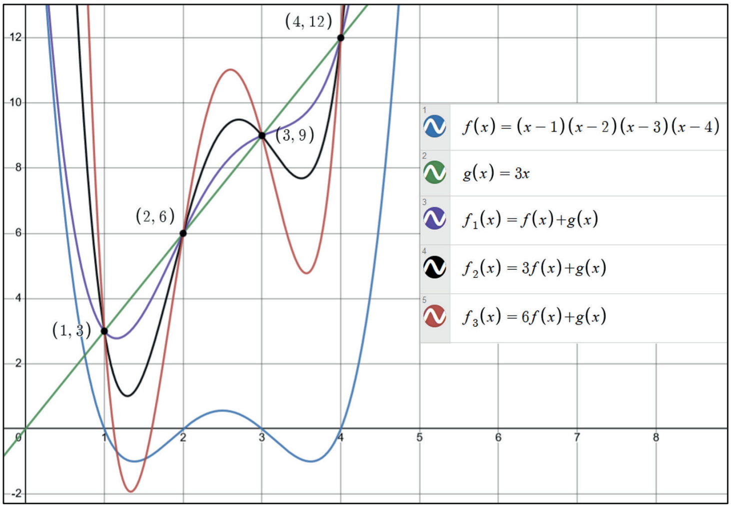

Figure 2. – Constructing a family of nonlinear polynomials.

Zazkis and Marmur (2021) argue that "the utility of advanced mathematical knowledge as a tool to instantly recognise student mistakes" (p. 96) is an important application of advanced mathematical knowledge to the work of teaching. The scripting task was a pivotal aspect of making this connection in that the motivating example of a "hidden" linear function was sourced directly from a dialogue. Furthermore, the challenge of finding a valid nonlinear polynomial to fit the required four points was directly relevant to the teachers because, as demonstrated in the scripts, they themselves had difficulty constructing such a function. In this way, the scripting task was a key aspect of the teachers' mathematical learning.

Exploring Further Connections Involving Functions

Because of the ubiquity of the function concept in both secondary and undergraduate mathematics, there are more opportunities to make connections than just those presented by Zazkis and Marmur (2018; 2021). In this section, we draw attention to other studies that have engaged learners in meaningful conversation on the function concept. Although these studies do not necessarily use scripting tasks to prompt such conversations, we propose ways that scripting tasks might naturally emerge from the existing mathematical groundwork.

Mirin et al. (2021) present two different definitions for function, each accepted as mathematically valid within the greater mathematics community, which nevertheless result in conflicting answers when questions are asked about certain functions. The authors motivate their study by first presenting a short dialogue between two students who are in disagreement about whether a diagram similar to the one in Figure 3 represents a function. The disagreement centers around whether a function $f: X \to Y$ must act on every element of the domain X or else if it is allowed to "ignore" some elements.

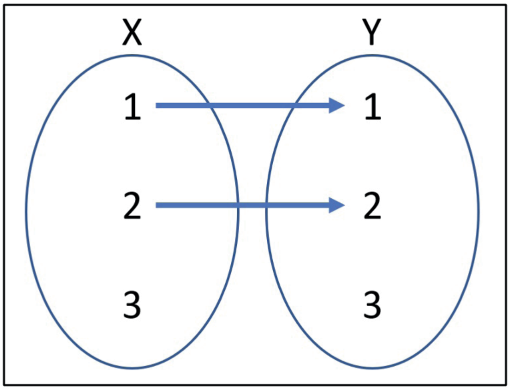

Figure 3. – Is the relationship pictured a function?

The argument captured by this initial dialogue in Mirin et al. (2021) could be translated almost directly into an interesting prompt for a scripting task. Just as in Zazkis and Marmur (2021), a hypothetical follow-up activity could focus on the diversity of definitions for function. However, Mirin et al. (ibid.) also illustrate that such a task could invite teachers to question the implications for choosing one definition over another. For example, in Halmos' (1960) definition of a function as a set of ordered pairs, the domain is the set that results from collecting all of the input values from these pairs and the codomain is the set that results from collecting all the output values; on the other hand, Bourbaki's (1968) definition describes the domain and codomain first, then characterizes a function as some subset of the cross-product. Under these definitions, proponents of Bourbaki's definition would recognize the relationship in Figure 3 as a function—whereas those who prefer Halmos' definition would not. Whether the domain comes before or after the specification of the function relationship also has a direct effect on when two functions should be considered "the same", and on what conditions are necessary for a function to have an inverse. In particular, does the condition for invertibility consist of injection and surjection or is injection alone sufficient?

This advanced perspective on the definition of function connects to school mathematics by illustrating how choice of definition can affect subsequent mathematical claims. Inverse functions, also taught in secondary mathematics, could be presented as an immediate consequence of the way functions are defined.

Functions and their inverses are also the topic of an undergraduate calculus lesson described in Burroughs et al. (2023). In that study, connections to teaching are presented in the form of "applications to teaching" problems such as pictured in Figure 4.

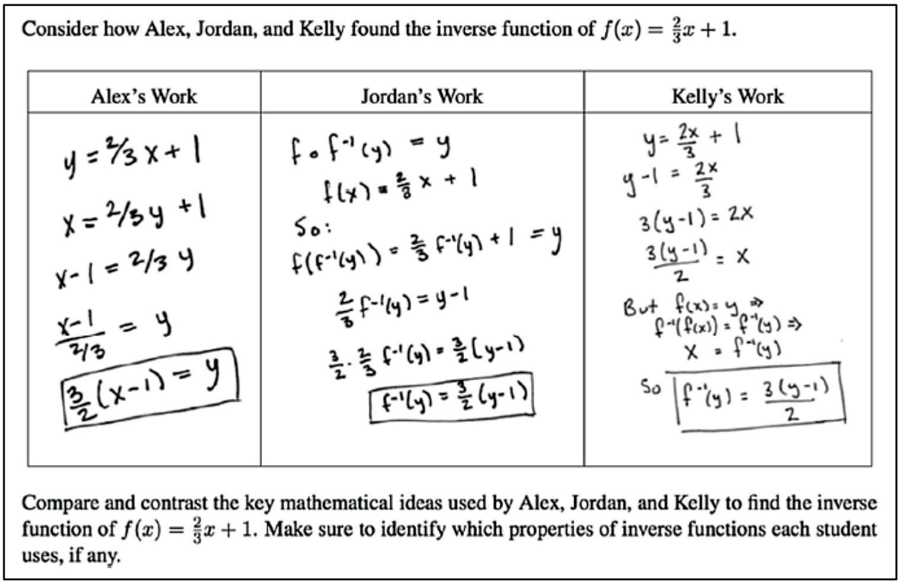

Figure 4. – Student work illustrating how to find the inverse of a function, originally in Burroughs et al. (2023).

Again, questions of domain and codomain arise and are used to motivate teachers to address the common method of "switching x and y, then solving for y" often presented in secondary mathematics classrooms as a way to find an inverse of a function. Burroughs et al. (2023) argue that the limitations of this approach have repercussions in future classrooms—such as when obtaining the derivative of the inverse trigonometric functions in a calculus course. Highlighting these limitations acts as a connection between secondary and undergraduate mathematics for the benefit of prospective teachers taking an introductory calculus course.

The task presented in Figure 4 could be translated into a scripting task by including a discussion amongst the students as they compare their methods for finding an inverse of a function. This dialogue could motivate teachers to consider the role that the definition of an inverse function played in Jordan and Kelly's work, and whether the three students in fact arrived at the same function. Like in previous studies, we note that this conversation would lay similar groundwork for a comparison of different definitions of function; however, the proposed scripting task might also facilitate discussion about a teacher's obligation to explain why the common methods that they teach are mathematically valid.

Extending scripting tasks to highlight connections

In this section, we follow Zazkis and Marmur's (2018; 2021) example by proposing how existing scripting tasks from recent literature can serve as a foundation for subsequent lessons that highlight connections between secondary and undergraduate mathematics. We elaborate on these connections, their value to prospective and practicing teachers, and suggest directions for activities that might leverage the content of student-generated dialogues to make these connections meaningful.

Star Polygons

Kercher et al. (2023) introduced the *scripting journey* as a modification of the typical scripting task. In a scripting journey, the scriptwriters construct an entire dialogue, without any initial prompt, based on their own experiences engaging with a novel mathematical setting. Kercher et al. (ibid.) consider the scripting journey to be a practical method of collecting data on students' activities during problem-solving tasks or mathematical investigations, supplementing other methods such as video recordings (e.g., Rott et al., 2021), problem-solving journals (Liljedahl, 2007), or portfolios (Gourdeau, 2019).

In Kercher et al. (2023), the participants' scripting journeys were inspired by their investigations of star polygons and their properties. A star polygon is an equiangular and equilateral polygon that is typically self-intersecting (see Figure 5 for examples) and is represented using a Schläfli symbol (n,k). Geometrically, this notation represents n vertices equally spaced around the circumference of a circle wherein every vertex p_i is connected to the vertex p_{i+k} (or p_{i+k-n}, when $i+k>n$) by a line segment—in this sense, k dictates the distance around the circumference one must "skip" in order to connect two vertices. Algebraically, star polygons can be associated with the notion of cyclic subgroups of $\mathbb{Z}$ under addition modulo n. That is, for any number of "skips" $k \in \mathbb{Z}$, the subgroup $k = \{ak | a \in \mathbb{Z}\}$ under addition modulo n is a star polygon. We note that, when $\text{GCD}(n,k)=1$, k is a generator of the group $\mathbb{Z}_\text{n}$. In the geometric interpretation, this corresponds to a unicursal connection joining every vertex.

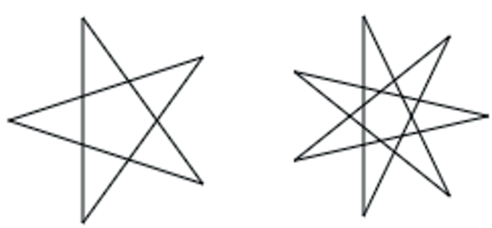

Figure 5. – A (5,2) and (7,3) star polygon.

Prospective and practicing teachers ($n = 27$) who engaged with the investigation of star polygons in Kercher et al.'s (2023) study were not provided with the above definitions; instead, they engaged with a star polygon task. In this task, working in groups, the teachers generated and refined their own definition and characterized the values of k that would result in a star polygon for some given number of vertices n. Afterwards, they wrote a scripting journey inspired by their group's shared progress. These scripting journeys exhibited a variety of *advancing mathematical activities* (in the sense of Rasmussen et al., 2015; Rasmussen, et al., 2005). An analysis of the scripting journeys revealed that the teachers communicated their understanding through symbols; extrapolated and refined definitions from the examination of examples; executed algorithms, which they then attempted to generalize to different or more abstract settings; and made, tested, and justified conjectures. In particular, Kercher et al. (2023) note the interdependence of these advancing mathematical activities—for example, student-characters in the scripting journeys constructed multiple star polygon symbols according to an agreed-upon algorithm, and in comparing these symbols, proposed definitions or made conjectures.

Although the teachers in Kercher et al.'s (2023) study do not attend to star polygons using the language of group theory, some student-characters in the scripting journeys do provide explanations that informally capture ideas from abstract algebra. Furthermore, some scripting journeys feature explicit reference to important concepts from number theory. It is from these observations that we anticipate one way of making connections between secondary and undergraduate mathematics.

Connections between Mathematical Disciplines

Star polygons are clearly geometrical in nature, and in exploring their properties, teachers are likely to reconsider their understanding of related topics familiar to geometry curricula in secondary schools. To illustrate: consideration of the $(5,1)$ star polygon calls into question whether regular polygons are in fact a subset of star polygons. This issue arose naturally in the scripting journeys of participants in Kercher et al. (2023). We propose that a follow-up lesson that builds on students' observations would facilitate teachers in assimilating star polygons into their existing geometry schemas. This practice is in line with Skemp's (1971) definition of understanding, and in this case, the understanding gained by teachers also allows them to connect school and undergraduate mathematics.

For example, if we accept regular polygons as a subset of star polygons, what familiar properties and theorems for regular polygons are then special cases of properties for star polygons—and what are the more general statements of these theorems? The sum of the interior angles of a regular polygon is often calculated in secondary schools using the formula $S_n = 180(n-2)$. Is this formula also applicable to star polygons? To answer this, undergraduate students must first recontextualize what they understand to be an "interior angle" for something like a $(5,2)$ star polygon (Figure 5). There is also a derivative relationship between the areas and perimeters of regular polygons (see Mamolo & Zazkis, 2012). Like the question of interior angles, deciding what to count as the perimeter and area of a self-intersecting shape is nontrivial. But once these terms have been clarified, does the same derivative relationship hold for star polygons?

Kercher et al. (2023) also note that they chose the star polygons task for their study in part because it allows for approaches that utilize multiple mathematical disciplines. For example, star polygons provide a geometric foundation for explorations in concepts from abstract algebra and number theory. One connection to the former subject has already been made in a preceding paragraph, but it can be extended. Consider the $(7,3)$ and $(7,4)$ star polygons; because $3+4 \equiv 0 \pmod 7$, these star polygons are the same shape. One can further justify this sameness by pointing to the cyclic subgroups <3> and <4> of $\mathbb{Z}$ under addition modulo 7 (i.e., the cyclic subgroups generated by 3 and 4, respectively) and noting that they are comprised of the same integers.

An interesting complication to this interpretation arises when the same shape appears on different numbers of vertices. For example, Kercher et al. (2023) reported that two separate scripting journeys represented the $(9,3)$ and $(12,4)$ star polygons as equilateral triangles, respectively. Neither group individually explored whether these star polygons were in fact the same, but a class discussion making this question explicit could lead to a follow-up lesson on group isomorphisms. That is, the cyclic subgroups of $\mathbb{Z}$ corresponding to these star polygons are clearly different: they are $\{0, 3, 6\}$ under addition modulo 9 and $\{0, 4, 8\}$ under addition modulo 12, respectively. But because the star polygons are the same shape, there is still something intuitively "the same" about these groups. Formalizing this intuition creates a powerful connection between disciplines and prompts students to search for isomorphism and group structure in unexpected places.

With respect to number theory, a key element of many of the scripting journeys reported on in Kercher et al. (2023) was relative primacy. Student-characters were particularly interested in finding values of k that were both less than and relatively prime to some given number of vertices n, and simultaneously, counting how many such k exist. This led to scripting journeys that partially rediscovered Euler's totient function, also called Euler's phi function—a function which provides exactly such a count and is signified by ϕ. The student-characters who began to recreate Euler's phi function only explored cases in which n could be decomposed into relatively few prime factors. An example of a meaningful task for a follow-up lesson would be to present teachers with a hypothetical student who wonders how many different values of n exist such that $\frac{n}{100}$ is a proper fraction in lowest terms. This situation introduces an application for Euler's phi function in the context of teaching, creating a practical connection to secondary mathematics. Deriving a version of the phi function that works for larger values (i.e., understanding how to calculate $\phi(100)$) could help teachers to grow the nascent ideas seen in the scripting journeys into general claims, engaging them in related questions of divisibility and combinatorics.

A Formal Language for Informal Ideas

Multiple strands of mathematics education research promote pedagogies that utilize student's ideas as the foundation for building mathematical theory—for example, realistic mathematics education (Gravemeijer & Doorman, 1999). For teachers who choose to employ one of these styles of instruction in their classrooms, a key skill to develop is the ability to help students convert their informal intuitions about mathematical situations into the formal mathematical register (see Zazkis, 2000). This act of translation might manifest as a notational challenge, in which students must capture the nuances of spoken language with mathematical symbols. Additionally, we perceive a parallel challenge: helping students to align their potentially informal standards for argumentation and justification with the more rigorous standards of mathematical proof-writing.

Teaching students how to formalize ideas, create and use notation, and recognize the strengths or limitations of a justification are not typical elements of a mathematics curriculum—particularly in secondary schools. In this practical sense, it is not clear that these skills are "mathematics content". We argue, however, that making connections between school mathematics and undergraduate mathematics transcends simply pointing to similarities between the mathematical content taught in both settings. Connections also include similarities in practice, an observation explored in more detail by Wasserman (2023a). Wasserman sought teachers' input on which practices they associate with mathematics and which they associate with pedagogy, hoping ultimately to identify a selection of mathematical practices that are simultaneously effective pedagogical tools. Once these *pedagogical mathematical practices* have been identified, calling attention to their use in an undergraduate mathematics classroom creates a connection to school mathematics.

But more effective than simply calling attention to these practices, we argue, is providing teachers with opportunities to personally engage with and reflect on these practices in their own mathematical work—such as when investigating star polygons and writing an accompanying scripting journey. As Kercher et al. (2023) observed, the dialogues written by teachers in these scripting journeys featured extensive use of symbolic notation and argumentation strategies. We believe that a follow-up lesson designed specifically to call attention to these aspects of mathematical practice would be better received by teachers as valuable pedagogical practices in light of their own recent experiences.

For example, in Wasserman's (2023a) study, one example of a pedagogical mathematical practice is using concrete examples to reason about a more general claim. This was a prominent strategy exemplified within the scripting journeys responding to Kercher et al.'s (2023) star polygon task. An effective follow-up task could ask teachers to generate examples from secondary mathematics in which specific examples play an important role in helping students understand a more general concept. Discussion could later attend to situations in which concrete examples are (and importantly, are not) sufficient evidence to prove a general mathematical claim.

Zero Divisors

Zazkis and Cook (2018) argued for integrating scripting tasks into a set of more conventional data collection strategies, such as conducting interviews, teaching experiments, and collecting students' written productions. The initial research of Cook (2014) identified students' difficulties with the Zero Product Property (ZPP), which states that, in an integral domain, if $ab=0$ then $a=0$ or $b=0$. The illustrative cases included students' over-generalizing the ZPP in a context where it is not applicable, such as when solving equations in a ring with zero divisors; conflating ZPP with its converse; and not taking advantage of the ZPP when an equation in $\mathbb{R}$ was presented in a factored form.

To further investigate students' understanding of the ZPP, Zazkis and Cook (2018) designed a scripting task, in which the authors presented a flawed proof that invoked the ZPP where it was not applicable (in a ring with zero divisors). They invited undergraduate students ($n=17$) to respond to this proof in the form of a dialogue between a teacher and her students, as if the proof were presented in class by a student. The scriptwriters were asked to identify the possibility for misconceptions or incomplete understanding as their characters worked on the presented proof. The scripts confirmed the findings of prior research regarding students' difficulties with the ZPP. Even when the flaw in the presented proof was correctly identified, it was identified by a teacher-character responding to a student error. The authors considered this to be an instance of the scriptwriter's reflection on a personal learning experience. The results also highlighted nuances in students' understanding of rings, such as the tendency to infer existence of inverses from the existence of an identity element. The scripts also provided insights into students' ideas about the notion of mathematical proof.

We foresee a professional development session with teachers that capitalizes on the research results presented above and aims at establishing explicit connections between school mathematics and undergraduate mathematics. As reiterated in research literature, such connections are not made by students without explicit instructional techniques, so they should be purposefully developed and articulated (Cuoco, 2001; Wasserman et al., 2017).

Our claim is that such a session may not only establish or strengthen the connections, but also extend teachers' understanding of the school mathematics they teach.

Attention to Basic Assumptions

According to Skemp (1971), "to understand something means to assimilate it into an appropriate schema" (p. 46). But how is it possible to understand better something that is already well understood? Zazkis (2008) contends that to understand something better means to assimilate it into a richer or more abstract schema.

Assimilating a known fact into a more advanced schema often includes an explicit acknowledgement of the conventional assumptions the fact carries and the possible generalizations it suggests. For example, the statement, "A whole number is divisible by ten if and only if the last digit (i.e., the units digit) is zero" is true under the implicit assumption that the number is represented in base ten. This assumption limits the scope of applicability of this divisibility test, but also invites variation and generalization: "a number represented in base 5 (or in base B) is divisible by 5 (or by B) if and only if the last digit (i.e., the units digit) is zero". So, a familiar property becomes a special case of a general property related to number representation.

We return here to the ZPP, which in school mathematics is presented explicitly as a valuable strategy in solving quadratic equations. That is, when a quadratic equation is written as a product of two factors, a reasonable and often preferred approach towards finding the solution set is to equate each factor to zero. This is an instantiation of the ZPP. However, as is usual in school mathematics, the basic assumption that limits the scope of this property—in this case, that the mathematical setting is the real numbers—is not mentioned explicitly. The same assumption is made when it becomes necessary to introduce the multiplicative inverse of a number, an idea often relevant to school mathematics (e.g., when multiplying by a reciprocal). In $\mathbb{R}$, every real number y, other than zero, has a multiplicative inverse y' so that $y \times y' = y' \times y = 1$. Zero is the only number that does not have a multiplicative inverse because no number multiplied by zero results in 1. This lack of multiplicative inverse also serves as one of the explanations for why division by zero is undefined. However, both the ZPP and the idea of multiplicative inverses are introduced well before students are aware of the field structure of the relevant number set.

Exposure to abstract algebra, and ring theory in particular, highlights the scope of applicability of the ZPP and the existence of multiplicative inverses. For example, in the ring $\mathbb{Z}_{12}$ – the set $\{0, 1, 2, \ldots, 11\}$ with addition and multiplication modulo 12 – the product of 3 and 8 results in zero. This is a pair of zero divisors; other zero divisor pairs in this ring are 3 and 4, 2 and 6, and 4 and 9. Further, zero divisors never have multiplicative inverses. Returning to $\mathbb{Z}_{12}$, for example, there is no number that when multiplied (modulo 12) by

3 results in 1. On the other hand, in $\mathbb{Z}_{12}$, the number 5 is its own multiplicative inverse as $25 \equiv 1(mod\,12)$. The same applies to the number 7, which when multiplied by itself results in $49 \equiv 1(mod\,12)$. As such, the idea of rings (that are not integral domains) invites teachers to reconsider the properties of zero as related to the ZPP and its uniqueness in terms of the non-existence of multiplicative inverses.

We believe that a lesson that explicitly explores these issues is more valuable as a follow up to a scripting activity, similar to the one discussed by Zazkis and Cook (2018), as it helps teachers reflect and acknowledge their "met-befores" that may have overshadowed their newly learned abstract algebra. Álvarez et al. (2022) describe the structure of such a lesson, intended for use in an undergraduate abstract algebra course, that connects to secondary mathematics by asking teachers to reconsider basic algebraic assumptions that in fact are only true because $\mathbb{R}$ is a field. Álvarez et al. (2022) motivate this conversation by featuring hypothetical student work, such as when one student incorrectly applies the ZPP in an inappropriate setting but arrives at a correct solution (Figure 6).

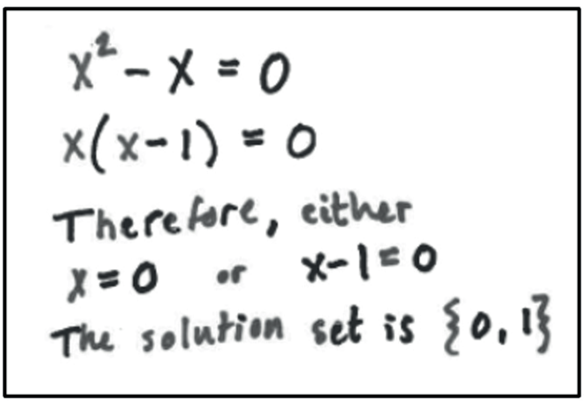

Figure 6. – Applying the ZPP in $\mathbb{Z}_4$

Conversations arising from considering this student's work can be extended to additional basic assumptions, such as geometry on a plane or base ten positional representation, that are not acknowledged explicitly unless breached.

The Many Faces of Multiplication

One important issue, which is implied but not highlighted in the work of Cook (2014) and Zazkis and Cook (2018), relates to the notion of multiplication and the multiplication symbol. In their brief overview of the related background, Zazkis and Cook note that "$3 \cdot_{12} 4 = 0$". That is, the symbol "$\cdot_{12}$" is carefully used by the authors to direct readers' attention to the fact that this multiplication is carried out modulo 12.

However, a few lines later they write “in $\mathbb{Z}_{12}$, 5 is invertible because $5 \cdot 5 = 1$”. Having established the applicable set in which the multiplication is performed, they choose not to emphasize that multiplication is performed modulo 12 and instead write the same symbol as is usually used when multiplication is performed with real numbers. This is a common trend in mathematics, where the meaning of a symbol depends on the context in which the symbol appears. That is, we call many operations “multiplication” and use the symbol for “familiar multiplication”, even when this familiar symbol and familiar word invite interpretations that may not be applicable in the given context.

We believe that this trend—which is an example of polysemy of the word “multiplication” and its symbol—is a source of many confusions and errors. Polysemy is a notion in linguistics that refers to words with different but related meanings. This is to distinguish from the notion of homonymy, which relates to words with different meanings but the same pronunciation and often the same spelling. While the meaning of homonyms is usually implied by the context, for example, in distinguishing a deposit in a *bank* from a steep *bank* of a river, polysemous words often appear in the same context, such as a mathematics lesson. Zazkis (1998) discussed the polysemy of the word quotient, which has different (but related) interpretations in the context of division of whole numbers and division of rational numbers. In considering 13 divided by 2, the whole number quotient is 6, but the quotient—as the result of rational number division—is 6.5. Note that in both cases, this division can be denoted by the same symbol: “$13 \div 2 = 6.5$” and “$13 \div 2 = 6$ remainder 1”.

Mamolo (2010) extended the notion of polysemy of words to that of symbols. As in the example above, the same symbol is used to indicate different kinds of division. Similarly, when we work in group theory, we use “+” to indicate addition, whereas the specific kind of addition may not be denoted explicitly. As a result, beginners in group theory mistakenly consider the group $(\mathbb{Z}_3, +)$ to be a subgroup of $(\mathbb{Z}_6, +)$, focusing on the subset relationship and not initially recognizing that the “+” symbol points to different operations.

A famous quote from Henri Poincaré is that “Mathematics is the art of giving the same name to different things”. As such, it is the labor of the teacher (of undergraduate mathematics) to direct learners’ attention to the different things called by the same name and denoted by the same symbol, seeking to capitalize on the differences and similarities. A possible task for teacher education is to compare “regular” multiplication to multiplication in $\mathbb{Z}_{12}$ and to identify similarities and differences. While the initial interpretation may involve listing the same results (such as $2 \cdot 3 = 6$ and $2 \cdot_{12} 3 = 6$) and different results (such as $5 \cdot 5 = 25$, versus $5 \cdot_{12} 5 = 1$), it should proceed to exploring relationships in the related multiplication tables, properties of zero and 1, and the existence of inverses. Such an exploration may also extend to exploring different meanings of multiplication—for example, when considered as a binary operation, multiplication exemplifies a function of two variables (see further Wasserman, 2023b).

Discussion and conclusion

We have demonstrated how engaging teachers with scriptwriting, either to envision interactions with students on a mathematical issue or to report on their mathematical investigations, can lead to follow-up instruction that strengthens the links between undergraduate and school mathematics. These links manifested in different ways, each with different pedagogical affordances. For example, Zazkis and Marmur (2021) demonstrated that knowledge of the fundamental theorem of algebra could allow teachers to quickly reason about an algebraically complex function suggested by a student. In Kercher et al. (2023), we identified opportunities for teachers to strengthen the mathematical practice of their students by amplifying mathematically normative symbol use and argumentation. Finally, in Zazkis and Cook (2018), knowledge of zero divisors leads to discussions about hidden assumptions within secondary mathematics. Each of these connections, and the consequent implications for teaching, was catalyzed by a different scripting task.

Obviously, the instructional activities described here can take place unrelated to the scripting tasks. However, we note explicit benefits when instruction follows scriptwriting. First, as a tool for data collection, writing a script allows students to prioritize the presentation and discussion of the most pivotal aspects of their mathematical understanding. This affordance is especially evident in scripting journeys, wherein the dialogue mirrors the most impactful episodes of the students' authentic experiences. Second, we suggest that the possibility to connect to a personal experience, including personal stumbles when initially considering an appropriate response to a student, makes the learning more memorable. As a result, stronger mathematical knowledge informs a more thoughtful pedagogical response, that in turn supports students' mathematics.

For example, Zazkis and Kontorovich (2016) explored reactions of teachers to a student's question regarding the exponent (-1). A student inquired whether this was the same symbol used to indicate the inverse of a function f, denoted f^{-1}, and the reciprocal of a number, as in 5^{-1}. The teachers were asked to write a script that started by addressing the student's question, "Have they ran out or symbols, or what?" Most participants highlighted the idea that the same symbol can be used for different purposes, and it is important to distinguish whether it applies to a number or a function. Only a few participants attempted to establish a connection between the two uses of the exponent (-1), pointing to the fact that in both cases it indicated "going backwards" or "doing the opposite". The follow-up instruction highlighted the mathematical idea of inverse with respect to operation, comparing the operation of multiplication of numbers to the operation of composition of function. While such a perspective is informed by a study of group theory, we claim that it can be accessible to learners without any mention of groups. As such, we believe that this experience equipped teachers with ways to respond to a student's curiosity regarding the exponent (-1), or possibly to prompt such curiosity.

Recent research on scriptwriting outlined multiple benefits for teachers, teacher educators and researchers. In particular, with the assumption that the scripts reflect scriptwriters' personal views, teachers' scripts provide researchers and teacher educators with information about teachers' perceptions of teaching. They also offer a glimpse into scriptwriters' mathematical knowledge and their pedagogical aspirations. In this paper we have highlighted an additional important benefit—scriptwriting by teachers serves as a springboard and catalyst for a subsequent exploration of connections between undergraduate and school mathematics.

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