

Symmetry as a Topic for the University Education of Pre-Service Teachers

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This article provides a comprehensive mathematical-didactic analysis of how the highly relevant topic symmetry can be prepared for the university education of PSTs. Methodologically, the analysis is embedded in a design research cycle and serves as preparation for the actual design of learning activities. The procedure of “specifying and structuring” learning objects is used and adapted in such a way that, in addition to mathematical aspects, profession-oriented references to school mathematics are also considered. An essential result of the analysis is the formulation of so-called interface aspects to symmetry, which prove to be helpful in establishing such references.

Keywords: symmetry, second discontinuity, interface aspects, specifying and structuring, design research, pre-service teachers

La simetría como tema para los profesores universitarios en prácticas

Este artículo proporciona un análisis matemático-didáctico exhaustivo de cómo el tema de la simetría, de gran relevancia, puede ser pre parado para la educación universitaria de los PST. Metodológicamente, el análisis se inserta en un ciclo de investigación de diseño y sirve como preparación para el diseño real de actividades de aprendizaje. El procedimiento de “especificación y estructuración” de los objetos de aprendizaje se utiliza y adapta de tal manera que, además de los aspectos matemáticos, también se tienen en cuenta las referencias a las matemáticas escolares orientadas a la profesión. Un resultado esencial del análisis es la formulación de los denominados aspectos de interfaz a la simetría, que resultan útiles para establecer dichas referencias.

Palabras-claves: simetría, segunda discontinuidad, aspectos de interfaz, especificación y estructuración, investigación sobre el diseño, profesores en prácticas

La symétrie comme sujet pour les enseignants en formation à l’université

Cet article fournit une analyse mathématique-didactique complète de la façon dont le sujet très pertinent de la symétrie peut être préparé pour l’enseignement universitaire des PST. D’un point de vue méthodologique, l’analyse s’inscrit dans un cycle de recherche sur la conception et sert de préparation à la conception réelle d’activités d’apprentissage. La procédure de « spécification et de structuration » des objets d’apprentissage est utilisée et adaptée de telle sorte que, outre les aspects mathématiques, les références aux mathématiques scolaires orientées vers la profession sont également prises en compte. Un résultat essentiel de l’analyse est la formulation de ce que l’on appelle les aspects d’interface de la symétrie, qui s’avèrent utiles pour établir de telles références.

Mots-Clés: symétrie, seconde discontinuité, aspects d’interface, spécification et structuration, recherche en design, enseignants en formation initiale.

I Introduction

Symmetry is a mathematical concept relevant from the first grade to current scientific mathematics (e.g., Dreyfus & Eisenberg, 1990; Franke & Reinhold, 2016; Weigand et al., 2018; Weyl, 2017). For mathematics, symmetry is repeatedly referred to as a fundamental idea (e.g., von der Bank, 2016) that is also important in research and teaching in other natural sciences, such as physics and chemistry (e.g., Duda et al., 2020; Livio, 2012). Moreover, symmetry is one of the mathematical concepts that people encounter daily, regardless of formal mathematical training in, for example, art, architecture, nature, or music (e.g., Hargittai & Hargittai, 1994). Accordingly, it is essential that pre-service mathematics teachers (PSTs) build a comprehensive understanding of the concept of symmetry and its intra- and extra-mathematical richness of relationships in their studies. In this paper, I explore how such a treatment can be designed and structured in the university part of teacher education. Thereby, the treatment of symmetry with the means of academic mathematics is intended to be explicitly in a *profession-oriented* way in which bridges are built to later teaching. This is intended to counteract the second discontinuity in teacher training, which Felix Klein described in the early 20th century (Klein, 2016). It is widely accepted and empirically supported (e.g., Buchholtz et al., 2013; Darling-Hammond, 2011; Hoth et al., 2020) that just teaching PSTs academic mathematics is not sufficient for overcoming the second discontinuity (the so-called Intellectual Trickle-Down Theory, Wu (2011)). As part of a design research project, we are using the example of a university course, “Geometry for PSTs,” to address the question of which innovations can be used to tackle this problem. The considerations on the concept of symmetry presented in this paper form an essential basis for this and provide a mathematics-mathematics-didactic networked perspective that also represents added value beyond our specific project as a theoretical basis for the teaching and learning of symmetry at school and university.

2 Background and aims of this paper

In this section, I provide an overview of the university course in which the considerations in this paper arose and briefly discuss the theoretical and methodological background to the implementation of profession orientation (Section 2.1). Building on this, I present the objectives of this paper (Section 2.2) and describe the methodological approach to achieving them (Section 2.3).

2.1 Context: The Course “Geometry for PSTs”

The considerations I describe in this paper are based on and extend the design research project for a university course, “Geometry for PSTs,” described in Hoffmann (2022). That course is taken by 6th-semester PSTs who will later teach at a German “Gymnasium”. In

Germany, the Gymnasium is the “highest level” school, ultimately leading to the Abitur, a final exam that must be taken to enter university studies in any subject. The project aimed to research and develop a concept for a course on plane Euclidean geometry attended only by PSTs, explicitly focusing on implementing the design principle of *profession orientation* (Hoffmann & Biehler, 2023, p. 739 f.). This means that learning opportunities are designed to combine academic mathematical knowledge with typical practices and tasks that are important in math-related situations in teaching. Those include, for example, analyzing and selecting approaches and tasks in textbooks as well as assessing and responding to students’ contributions (c.f., Ball & Bass, 2002; Prediger, 2013).

Two innovations are central to implementing *profession orientation* in the course “Geometry for PSTs.” First, two weeks of the course were used as so-called *interface weeks*. In those two weeks, two essential geometry concepts (congruence and symmetry) were treated not only in terms of academic mathematics but also under the consideration of profession-related aspects to create a solid foundation for both subsequent subject didactic courses and for future classroom practice.

Secondly, there was a so-called *interface ePortfolio* (the “e” comes from the fact that it was an electronic portfolio) that the PSTs had to work on regularly throughout the semester. The types of portfolio activities included so-called *interface tasks*, in which the PSTs had to use their academic knowledge in the context of (fictitious) situations that are typical for a teacher’s every-day work. Further details on using *interface ePortfolios* to implement professional orientation can be found in Hoffmann & Biehler (2023).

2.2 Developing a Theory-Based Structure for Teaching Symmetry to PSTs

As already described, symmetry is one of the geometric concepts for which *profession-oriented* learning opportunities were implemented in the “Geometry for PSTs” course. Substantial mathematical-didactic analyses are necessary for an appropriate theory-based construction of the lecture content (especially in the relevant interface week), the exercises, and the activities for the *interface ePortfolio*. The presentation of these analyses is the core subject of this paper. To this end, I address the following research questions:

(RQ 1.) What should PSTs learn about the concept of symmetry in a profession-oriented mathematics course at university?

(RQ 2.) What is a possible structuring of the content identified in answering RQ1?

RQ1 is formulated in very general terms and requires further characterization. First, the context of the university course means that “learning” is about academic mathematical

knowledge and skills relating to the concept of symmetry. The *profession orientation* then goes hand in hand with the fact that this rigorous academic perspective must be linkable to school mathematics teaching. In addition, the PSTs should be able to use these established links productively to cope with typical teaching situations.

Theoretically, this complements the acquisition of academic mathematical knowledge and skills by explicitly addressing facets of content-specific teacher expertise (Prediger, 2019, p.370). In this framework, the above-mentioned mathematics-related situations in teaching correspond to so-called “jobs”. Through profession-oriented learning opportunities, the PSTs’ repertoire of “practices” and “pedagogical tools” (also relevant constructs of content-specific teacher expertise) is strengthened for dealing with the “jobs”. Furthermore, explicit reflection assignments (c.f. Hoffmann & Biehler, 2023) also address the prospective teachers’ “orientations” about the relevance and usefulness of academic mathematics for teaching. All of this takes place in the context of a mathematics course in a fictitious setting from a predominantly mathematical perspective. In Hoffmann and Biehler (2024), we present a suggestion for a theoretical clarification of this mono-perspectivity. The orientation towards “jobs” and thus plausible situations from the everyday life of a mathematics teacher is an important design element of profession-oriented learning opportunities for PSTs in a broader sense, which is also used in other projects (e.g., Wasserman et al., 2017).

Looking at the existing mathematics didactic literature on the concept of symmetry, one finds that there are many contributions dealing with specific issues in learning symmetry in school (e.g., Aktaş & Ünlü, 2017; Dello Iacono & Ferrara Dentice, 2022; Dreyfus & Eisenberg, 1990, 1998; Duda et al., 2020; Götz et al., 2020; Hoyles & Healy, 1997; Leikin et al., 2000; Ng & Sinclair, 2015; Son, 2006; Tatar et al., 2014). Although some of them refer to mathematical background theories (e.g., Franke & Reinhold, 2016; Weigand et al., 2018), I could not find any papers in which the reference to the teachers’ or PSTs’ mathematical content knowledge was explicitly made, and in which systematic linkages between rigorous academic mathematics and professionally relevant requirements were implemented in the sense of counteracting the second discontinuity. By answering the two aforementioned research questions, I aim to contribute to closing this gap.

2.3 Methodological Approach: Specifying and Structuring in Topic-Specific Design Research

Methodically, I follow the tradition of German “Stoffdidaktik” using the modern framework of topic-specific design research (Prediger & Zwetzschler, 2013). At its core is a cyclical approach that consists of the following four steps: Specifying and structuring learning goals and contents (Step 1), (re)designing learning activities (Step 2), using and

researching the learning activities (Step 3), and developing and refining (local) theories (Step 4). In this model, the analysis of the mathematical topic on which profession-oriented learning activities are to be developed corresponds to Step 1 (specifying and structuring). Obviously, the results of this step contribute to answering the research questions. For the procedure of specifying and structuring, Hußmann and Prediger (2016) propose a “four-level approach”:

- *formal level*: “addressing the mathematical objects and phenomena in their formal presentation and their logical structure”
- *semantic level*: “addressing sense and meanings – e.g., by big ideas and basic mental models – of the mathematical topic to be learned and epistemological aspects of the structure between them”
- *concrete level*: “addressing the realization of the teaching-learning arrangement by core ideas, problems, and situations, in which the mathematical knowledge is relevant and could be constructed in a generic way”
- *empirical level*: “addressing cognitive and possibly social aspects of student thinking, typical resources, pathways, and obstacles”

(Hußmann & Prediger, 2016, pp. 35,37)

In this paper, I thoroughly discuss the formal level (Section 3) and the semantic level (Section 4)¹. These two levels are relevant foundations for further theoretical and empirical considerations and provide enough potential for an entire paper. For the other two levels, I provide an overview and outlook on relevant aspects that must be dealt with in another paper (Section 5).

A particular challenge in the context of this paper is that specifying and structuring refers not only to the mathematical learning object (*symmetry*) but also to its profession-oriented framing. In particular, this means that the role of the concept of symmetry in school mathematics must be taken into account in addition to the treatment with the means of academic mathematics. Therefore, extending the guiding questions for the four levels is necessary. I present the corresponding considerations at the beginning of the relevant sections of the analysis, always describing first the procedure proposed by Hußmann and Prediger (2016) and then elaborating and theoretically locating necessary adaptations.

1. As already mentioned, this paper is part of a project on geometry education of PSTs that was initially presented in the PhD thesis Hoffmann (2022) and is now continued in different directions. Parts of the contents of the following analysis are already part of the thesis (pp. 231–254) in a modified form but are published here for the first time in English. For reasons of readability, the corresponding text passages are not pointed out each time separately.

I would also like to point out the special role of the formal level: In the school context, this level provides a mathematical background that is not treated with such rigor and systematicity in the classroom. However, university mathematics courses cover this topic with precisely this exactness and rigor.

It is important to note that the following explanations and analyses are not the results of a recipe-like approach but contain constructive and creative elements. Hußmann and Prediger (2016, p. 37) formulate in this regard:

“The fact that nearly none of these questions has a simple and unique answer already gives us the first hint that the four-level approach cannot offer a simple method with recipes to securely reach a good specification and structure. But the questions can support a complex process and the navigation between the four levels allows for a higher transparency within an important area of didactical work.”

I take up this aspect again separately in the discussion section.

3 The Formal Level: Mathematical Background to the Concept of Symmetry

The description of the formal level of symmetry in this section is closely linked to the content of the course “Geometry for PSTs” and the courses usually attended by PSTs at Paderborn University beforehand. In order to make the following analyses applicable to other situations, I would like to make the anticipated prior knowledge explicit here:

For treating symmetry in the following sense, basic knowledge of essential mathematical concepts such as sets, mappings (incl. injectivity, surjectivity), and groups (incl. subgroups) is essential. At Paderborn University, these are usually acquired in first-year analysis and linear algebra courses. In addition, the concept of isometry and some theory of Euclidean or neutral geometry is required. The specific axiom system is not essential; the content could, for example, also be embedded in a non-synthetic approach to geometry that works with the methods of linear algebra.

3.1 Guiding Questions

Hußmann and Prediger (2016, p. 35) characterize the formal level by “addressing the mathematical objects and phenomena in their formal presentation and their logical structure” and formulate the following five guiding questions (p. 36):

(F1) Which concepts and theorems have to be acquired? (*Specifying*)

(F2) Which procedures have to be acquired, and how are they justified formally? (*Specifying*)

(F3) How can the concepts, theorems, justifications, and procedures be structured in logical trajectories? (*Structuring*)

(F4) Which connections are crucial, which are contingent? (*Structuring*)

(F5) How can the network between concepts, theorems, justifications, and procedures be elaborated? (*Structuring*)

The guiding questions must be considered in light of the fact that the original approach was designed for the school context. There, the formal level has the role of a kind of background theory, which, however, is not made explicit in school mathematics lessons (Hußmann & Prediger, 2016, p. 56). This is different for design in the context of university courses since there, all mathematical concepts and phenomena are discussed in terms of the axiomatic method of academic mathematics in a deductively structured theory building. The formal level is, therefore, not a background theory but comprises the content of academic mathematics that the learners (here PSTs) deal with directly.² Accordingly, structuring at the formal level can only be discussed in light of the underlying theoretical structure, like the axiom system used, already defined concepts, introduced notations, and already proven theorems. At the formal level, this raises another guiding question:

(F0) How can the focused concept and associated properties and phenomena be formalized in the axiomatic-deductive structure chosen for the learning context? (*Specifying*)

To create a reference to the future teaching profession at the formal level, the selection of concepts, theorems, and procedures (F1, F2) must be aligned with the content of school teaching. Of course, this does not prevent other aspects from being considered based on additional objectives. Additionally, in preparation for the analyses on the semantic level, it can be helpful to discuss alternative approaches to investigate the dependence of certain properties and phenomena on a specific theoretical foundation.

3.2 Specifying Symmetry on the Formal Level

In the course Geometry for PSTs, we build on an axiomatic approach by Iversen (1992) to formalize plane geometry (Hoffmann, 2022; Hoffmann, Hilgert, Weich, 2024). A metric space (Definition A.1)³ is taken as a basis, and the so-called incidence axiom (Axiom

2. Of course, when specifying and structuring university learning objects, it can also happen that theoretical perspectives are adopted at the formal level that are not discussed with the university students in the same way later in the learning activities. However, this is not the case in the context presented here: Symmetry is treated at the mathematical level in the course “Geometry for PSTs” in the same rigorous way as I present it in the section on the formal level.

3. All cited definitions, theorems, etc. whose numbering starts with “A” can be found in Appendix A of this paper.

A.4) and the reflection axiom (Axiom A.5) are introduced on top of it. In this way, one receives an axiomatization of the neutral plane geometry (Definition A.6), which can be restricted then by the well-known parallel axiom (Axiom A.8) to the Euclidean geometry (Definition A.9).

In this setting, the symmetries of a figure (as a subset of the surrounding metric space) are usually described as the bijective isometries (Definition A.3)⁴ that leave the figure invariant. Exactly in this sense, the symmetry group of a figure is defined as in Definition 3.1.

Definition 3.1 (Symmetry Group): *Let (X, d) be a neutral Plane and $F \subset X$ an arbitrary figure (subset). Then we define the symmetry group of F as $\text{Sym}(F) := \{\varphi \in \text{Isom}(X) \mid \varphi(F) = F\}$.*

Of course, symmetries in school mathematics are not considered in the context of general metric spaces. However, a restriction to \mathbb{R}^2 and the related specialization of the general isometry notion does not simplify the formalism. A more general formalization of symmetry on more abstract sets endowed with structures can be done via general automorphisms (e.g., Weyl, 2017, p. 136). From the perspective of profession-orientation, however, in this approach, there is a danger that the relations to Euclidean plane geometry are lost in abstract formalism, and the concepts learned are more difficult to transfer to the teaching of geometry.

In Definition 3.1, just as in many other textbooks on elementary geometry, the set of symmetries of a figure is defined by the symmetries $\text{Isom}(X)$ of the underlying metric space. Concretely, this means that figures in the Euclidean plane can have reflectional symmetries, rotational symmetries, translational symmetries, and glide-reflectional symmetries because every nontrivial isometry of the Euclidean plane is a reflection, a rotation, a translation, or a nontrivial glide reflection. Other symmetries are not allowed by the chosen definition of symmetry groups. However, apriori, it is not clear whether there cannot be somehow subsets and bijective isometries defined on them which cannot be continued to isometries on the whole metric space. Indeed, Example 3.2 shows that exactly this phenomenon can occur in general metric spaces as soon as one does not require the axioms of the neutral plane.

Example 3.2: *Let $F := \{1, 2\}$ be a subset in the metric space $(\mathbb{N}, |\cdot|)$. Then F has two symmetries, namely the identity and the isometry, which interchanges 1 and 2. However, the interchange of 1 and 2 cannot be continued to an isometry of \mathbb{N} : If there were such an isometry $\varphi: \mathbb{N} \rightarrow \mathbb{N}$ with $\varphi(1) = 2$ and $\varphi(2) = 1$, $\varphi(3)$ must be ≥ 2 because isometries are injective. But then it follows that $|3 - 2| = 1 \leq 2 \leq |\varphi(3) - 1| = |\varphi(3) - \varphi(2)|$, so φ cannot be an isometry. Contradiction.*

4. In general metric spaces, isometries are always injective due to positive definiteness (Definition A.1 (M1)). However, the example $\varphi: \mathbb{N} \rightarrow \mathbb{N}, n \mapsto n+1$, as an isometry in the metric space $(\mathbb{N}, |\cdot|)$, disproves surjectivity because $1 \in \mathbb{N}$ has no preimage.

However, this continuation property is true in neutral planes. After finding no proof in the literature (which uses elementary means), we have developed a proof using elementary geometric arguments.

Theorem 3.3 (Continuability of isometrics in neutral planes): *Let (X, d) be a neutral plane and $F \subset X$ a figure. Then every isometry $F \rightarrow F$ is continuable to an isometry $X \rightarrow X$.*

Proof. See Hoffmann et al. (2024, p. 229 f.). \square

Thus, figures of the Euclidean plane can have precisely the symmetries mentioned above. The next step of structuring is to justify the notion *symmetry group*.

Theorem 3.4 (Group Property of the Symmetry Group): *Let (X, d) be a neutral plane and $F \subset X$ an arbitrary figure. Then $(Sym(F), \circ)$ is a group.*

Proof. $(Sym(F), \circ)$ is a subgroup of $(Isom(X), \circ)$: If $\varphi, \psi \in Sym(F)$, then due to $\varphi(\psi(F)) = \varphi(F) = F$ and the bijectivity of φ also $\varphi \circ \psi$ and φ^{-1} are elements of $Sym(F)$.

That $Sym(F)$ is a group concerning the concatenation of mappings provides a structure and associated logical dependencies between different symmetries of a figure. Thus, the existence of certain symmetries implies the existence of other symmetries. Important examples in the context of the Euclidean plane, which are also relevant for school geometry, are summarized in Theorem 3.5.

Theorem 3.5 (Dependency properties of rotational symmetries and reflection symmetries). *Let (X, d) be a neutral plane and $F \subset X$ be an arbitrary figure. Then the following statements hold:*

(i) *Let $\alpha, \beta \in [0, 2\pi[$ and $Z \in X$. If F is α - and β -rotationally symmetric concerning Z , then it is also $(k\alpha + l\beta \bmod 2\pi)$ -rotationally symmetric concerning Z for all $k, l \in \mathbb{N}_0$.*

(ii) *If there is a smallest angle $\alpha_{min} > 0$ such that F is α_{min} -rotationally symmetric, then $\alpha_{min} = 2\pi k$ with $k \in \mathbb{N}$.*

(iii) *There is a possibility that there is no such smallest angle*

(iv) *If F is reflectional symmetric with respect to two lines $g, h \subset X$ with $g \cap h = \{P\}$ ($P \in X$), then F is also $\pm 2\alpha$ -rotational symmetric concerning P . Here α is the angle between g and h .*

Proof. Use the properties of rotations, reflections, and their concatenations. \square

Especially in the lower school grades, only symmetries of bounded figures (in the metric sense) are considered. Theorem 3.6 provides that these can indeed only be reflection and/or rotational symmetric; if a symmetry group of a figure (in an Euclidean Plane) contains translations or glide reflections, the figure is automatically unbounded.

Theorem 3.6 (Symmetry Properties of Bounded Figures): *Let F be a figure in a Euclidean plane. Then all mappings $\varphi \in \text{Sym}(F)$ have a fixed point. In particular, F can only be reflectional and/or rotationally symmetric.*

Proof. See Hoffmann et al. (2024, p. 235). \square

Many activities in school geometry on the topic of symmetry are about completing symmetrical figures. The associated mathematical background can be formalized by the concepts orbit (Definition 3.7) and fundamental domain (Definition 3.8, Theorem 3.8).

The idea of the *orbit* is that for a point of a figure, all images of this point are united under the elements of the symmetry group (Figure 1).

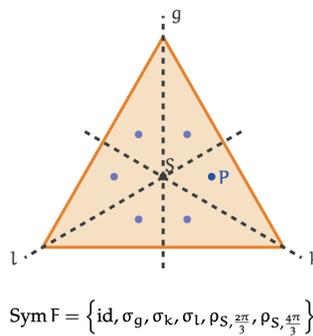


Figure 1. –The orbit of a point P in an equilateral triangle.

Definition 3.7 (Orbit) *Let F be a figure in a neutral plane (X, d) . Then we define for $P \in F$ by*

$$\text{Sym}(F) \cdot P := \{\varphi(P) \mid \varphi \in \text{Sym}(F)\}$$

the orbit of P .

The concept of the orbit makes it clear that in a figure with known symmetries, some information is, to a certain extent, redundant. To uniquely specify a symmetric figure, it is sufficient to know a subset that contains exactly one point from the orbit of each point in the figure. The other points of the figure are then obtained by applying all elements of the symmetry group to this subset. Subsets of this kind, which are minimal, are usually called fundamental domains (Figure 2).

Definition 3.8 (Fundamental Domain): *Let F be a figure in a neutral plane (X, d) . Then we call a set $B \subset F$ a fundamental domain of F if for each $P \in F$ the set $(\text{Sym}(F) \cdot P) \cap B$ contains exactly one point.*

Theorem 3.9: *In the situation from Definition 3.8, F can be reconstructed from any fundamental domain B .*

Proof.

$$F = \{\varphi(P) \in X \mid \varphi \in \text{Sym}(F) \text{ und } P \in B\} = \bigcup_{P \in B} \text{Sym}(F) \cdot P$$

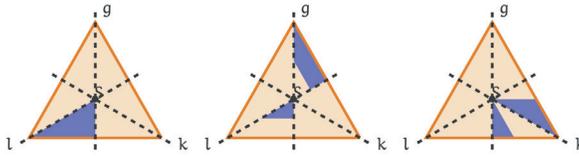


Figure 2. – Visualization of different fundamental domains of the equilateral triangle from Figure 1. In each case, there is exactly one point from the orbit of each point in the subsets marked in blue.

The definition of the fundamental domain is relational and not constructive. It provides a way to check whether a fundamental domain has been found but no procedure for how to find one. In fact, fundamental domains are not uniquely determined, but there are an infinite number of them for many figures.

It is also not trivial to prove that you have actually found all symmetries for a given figure. Of course, one can adopt a structure-theoretic perspective and - at least in the case of a regular n-gon - argue via the dihedral groups. A more elementary, more combinatorial alternative is shown in Example 3.10.

Example 3.10 (Completeness of symmetry groups): *We show that the isometries given in Figure 3 are all elements of the symmetry group of a square and thus all symmetries are indeed found.*

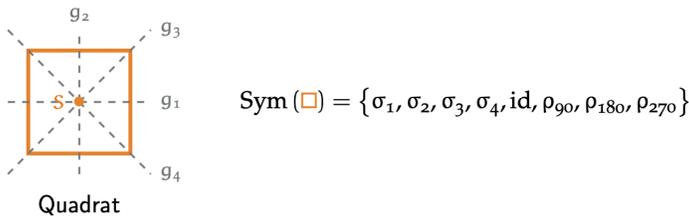


Figure 3. – Symmetry group of a square.

Corners must be mapped to corners (distance to the center of the square must be preserved under isometry). The square has four corners. There is, therefore, a maximum of $4! = 4 \cdot 3 \cdot 2 = 24$ possibilities. However, it is not possible for exactly one corner to remain fixed. In this case, a neighboring corner would become an opposite corner, but this cannot be achieved with isometry. This eliminates 2 possibilities for each corner (number of possibilities to permute three numbers without a fixed point), i.e., a total of 8 possibilities. A maximum of 16 possibilities remains. Furthermore, an isometry cannot leave two neighboring corners fixed but swap the other two corners. This eliminates a further 4 possibilities (number of neighboring pairs). A maximum of 12 possibilities remains. Finally, an isometric cannot move two neighboring corners by two positions and change the order of the other two corners. This eliminates a further 4 possibilities (number of neighboring pairs). Exactly 8 possibilities remain. This

corresponds to the number of symmetries found in the list, which is, therefore, complete. In retrospect, we have excluded all possibilities in the argument in which opposite points become neighbors.

3.3 Synthesis: First Thoughts on Structuring Symmetry

In Section 3.2, following the procedure described in Section 2.3, I presented a specification of the topic symmetry on the formal level. Accordingly (F0, Section 3.1.1), the complete theory construction was done on the neutral level axiom system used in the course Geometry for PSTs (Appendix A). The selection of definitions and theorems presented is oriented towards a mathematical foundation of symmetry considerations in school. In the sense of the guiding questions (F1) and (F2), the following theory elements were treated:

- Definition of symmetries of a figure using a formalization of “distances” (Definition 3.1)
- Statements about the fundus of possible symmetries of figures depending on the surrounding metric space (Example 3.2, Theorem 3.3) and the boundedness of the figure itself (Theorem 3.6)
- Statements about interrelationships between symmetries of a figure (Theorem 3.4, Theorem 3.5)
- Formalizations for the description for reduction and reconstruction of symmetric figures by non-symmetric subsets (Definition 3.7, Definition 3.8, Theorem 3.9)
- Rationale for justifying the completeness of a symmetry group (Example 3.10)

As essential mathematical prerequisites, besides basics on sets, functions, and metric spaces, a classification of the isometries of the Euclidean plane, as well as their concatenations, is required. Here, an alternative to the chosen approach appears: Instead of general metric spaces, the theory elements can also be easily transferred to the special case \mathbb{R}^2 with the Euclidean metric.

This raises the question of whether the restriction of general neutral planes to Euclidean planes makes a difference concerning the treatment of symmetry. Indeed, in general neutral planes, in addition to the symmetries known from Euclidean geometry, there is also the possibility of so-called *horolational symmetries*. Horolations are another type of isometries that arise from the concatenation of reflections on straight lines that are parallel but have no common perpendicular line. This case can only occur in non-Euclidean neutral planes (hyperbolic geometry).

The specification described on the formal level results in a first draft for structuring the learning object symmetry. The structure is based canonically on the deductive sequence of definitions and theorems and can be described by the following steps.

1. Which symmetries are possible?
2. Exploring the inner structure of symmetries of a figure with a focus on reflectional and rotational symmetries
3. Orbits and fundamental domains as formalization for “interesting subsets” of symmetric figures
4. Further, optional in-depth topics

For the last aspect, for example, the investigation of symmetries of unbounded figures of the Euclidean plane lends itself. The complete classification of the seven frieze groups and 17 wallpaper groups can be done on the described theoretical basis. Thereby the exploitation of the group properties (Theorem 3.4) and knowledge about the concatenation of Euclidean isometries is central. Another possible deepening consists in the treatment of symmetries of the space. However, this requires first an isometry theory of the three-dimensional Euclidean space.

Related to the two research questions, this section provides relevant concepts and connections that should be rigorously addressed in a math course for PSTs to provide a solid background for dealing with symmetry in school mathematics.

This first structuring is further substantiated in the next section by adding a profession-oriented semantic perspective. It becomes clear that this formal level represents a functional background for this.

4 The Semantic Level: Central Aspects of Symmetry Connected with School Mathematics

4.1 Guiding Questions

The core of the semantic level is given by Hußmann and Prediger (2016, p. 37) as “addressing sense and meanings – e.g., by big ideas and basic mental models – of the mathematical topic to be learned and epistemological aspects of the structure between them”. They formulate the following guiding questions (p. 36):

(S1) What are the underlying big ideas behind the concepts, theorems and procedures?
(*Specifying*)

(S2) Which basic mental models and (graphical, verbal, numerical, and algebraic) representations are crucial for constructing meaning? (*Specifying*)

(S3) How can the meanings be successively constructed by horizontal mathematization in the intended learning trajectories? (*Structuring*)

(S4) Which trajectories of vertical mathematization have to be elicited in order to initiate the invention/discovery of core ideas, concepts, theorems, and procedures? (*Structuring*)

(S5) How can the intended learning trajectories be sequenced with respect to the logical structure? (*Structuring*)

As with the formal level, the guiding questions are originally intended for the school context. Regarding the theoretical didactical concepts mentioned in (S1) and (S2), there are currently few insights in the field of higher education didactics. Therefore, the level of detail in this paper is not as high as it would be for a topic of school mathematics that has already been studied many times. Furthermore, also on this level, the guiding questions are to be supplemented to address the concern of implementing *profession orientation*:

(S6) What is the relevance of the mathematical learning object for teaching mathematics in school? (*Specifying*)

(S7) What role do the semantic structural elements of the learning object identified in (S1) and (S2) play for the school mathematics reference points elaborated in (S6)? (*Specifying*)

4.2 Specifying Symmetry on the Semantic Level – Part I: Mathematical Perspective

In the treatment of the concept of symmetry, as described in Section 3.2, three central ideas of modern mathematics play a role above all: (1) the set-theoretical embedding of the subject, which makes it possible to formalize an everyday visually perceptible phenomenon rigorously; (2) the mapping-geometric approach, which provides an efficient technology for defining and reasoning; (3) the basic mathematical approach of defining and studying structures.

Both these ideas and the central elements of the mathematical theory of the symmetry concept, as described in Section 3, are independent of any particular axiom system (e.g., the one used in the course *Geometry for PSTs*). Especially for the transferability of the mathematical knowledge learned in the course to the school context, a detachment from a particular axiomatic structure (which is in some sense legitimate but also always exemplary) is necessary. Precisely from this idea, so-called *interface aspects* to the concept of symmetry were elaborated in Hoffmann (2022, p. 239 ff.). The general idea behind *interface aspects* to academic mathematical concepts is described as follows.

“These [interface aspects] are the result of inductive mathematical didactic analyses of approaches (in the sense of definitional possibilities) to and properties (in the sense of central mathematical statements) of a mathematical concept, with particular attention to the role that the concept plays in mathematics education. In this way, interface aspects

represent a systematization of a mathematical theory, reflected from a profession-oriented point of view, that detaches subject-specific characterizations and important properties from a specific axiom system.” (Hoffmann, 2022, p. 152, translated)

According to this approach, the *aspect of invariance*, the *aspect of reconstruction and reduction*, and the *aspect of groups* have been described for *symmetry*:

4.2.1 Aspect of Invariance

In the central definition of the symmetry group (Definition 3.1), symmetries are understood as the different possibilities to map a figure bijectively (by isometries) onto itself. Under these isometries, the figure is *invariant*. I call this perspective the *aspect of invariance* (Hoffmann, 2022, p. 240).

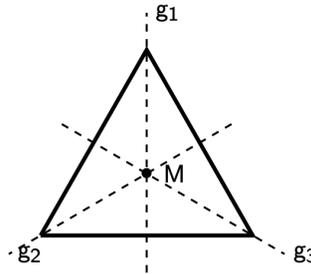


Figure 4. –The symmetries of the equilateral triangle result from the isometries (especially rotations and reflections), which leave it invariant

In this approach, symmetries of the equilateral triangle in Figure 4 can be described as follows: The triangle is reflectional symmetric with respect to the line g_1 because it remains *invariant* under the reflection at g_1 . It is also 120° -rotationally symmetric with respect to the point M because it remains invariant under the 120° -rotation around M . Of course, the invariance is usually not pointwise but as a whole set.

4.2.2 Aspect of Reconstruction and Reduction

The notion of *fundamental domains* described in Definition 3.8 justifies the approach to describe symmetries of a figure via the possibilities of reconstructing it from a subset of itself using the symmetries (Theorem 3.9). A figure is symmetric if it is composed of several “equal parts”⁵ which can be transformed into each other by the mappings of

5. The term is deliberately placed in quotation marks here. From a rigorous mathematical perspective, it would be wrong to speak of a decomposition into “congruent parts”. This is because, even if it looks like this from a preformal perspective, a symmetrical trapezoid, for example, cannot be divided into two “congruent

the symmetry group. Candidates for such a subset, which is minimal with respect to its information content, are described by the term *fundamental domain*. Note that this itself contains no information about how the reconstruction is performed. Thus, a set can be a fundamental domain for different figures with different symmetries (Figure 5).

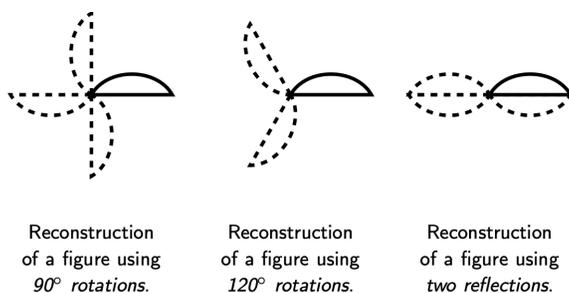


Figure 5. – Reconstruction of different symmetric figures from the same fundamental domain.

The symmetries of the figures are specified by the isometries necessary to reconstruct the figure. If the symmetry group is finite, a construction method can be specified: The elements of the symmetry group are first applied to the fundamental domain, and the union of the images with the fundamental domain yields a new figure, to which the isometries are again applied. This continues until reapplying all isometries no longer results in new points (Figure 6). The basis of this brute-force procedure is the proof of Theorem 3.9. However, in practice, this can be shortened considerably by using knowledge about the concatenation of isometries.

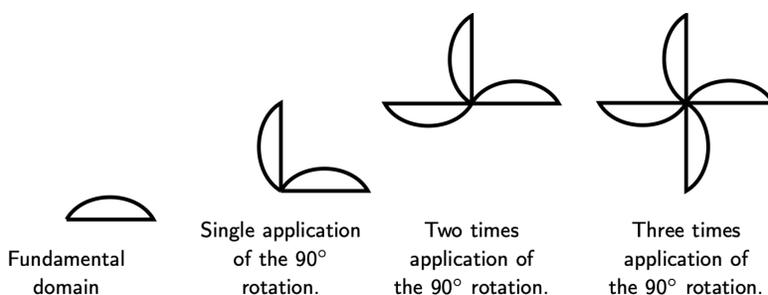


Figure 6. – Reconstruction of a 90° -rotationally symmetric figure from a fundamental domain. Any further application of 90° -rotation leaves the figure invariant.

halves” from a set-theoretical perspective, as the points lying in the trapezoid on the axis of reflection must also be assigned to one of the halves. However, the bijectivity is then lost, and the two halves are optically but not set-theoretically congruent.

If one turns the described point of view around, one finds that with the possibility of the reconstruction inversely also the potential for the reduction comes along. The complexity of the overall figure can be reduced to a fundamental domain together with the symmetry mappings. I call this perspective on symmetry the *aspect of reconstruction and reduction* (Hoffmann, 2022, pp. 240 f.).

Behind the method of *reconstruction* described above, there is also hidden the possibility of purposeful *construction* of figures with certain symmetries in the following way: From an initial figure, one can *construct* a figure with desired symmetries by repeatedly applying the geometric mappings associated with the intended symmetries and uniting the resulting images.

4.2.3 Aspect of Groups

We have shown (Theorem 3.1) that the mappings describing the symmetry of a figure, together with the usual concatenation of mappings, form a group. On the one hand, this means that one can “calculate” with symmetries. In addition, using the group structure, the existence of certain symmetries can be used to infer the validity of other symmetries. I call this important property of the concept of symmetry the *aspect of groups* (Hoffmann, 2022, pp. 241 f.). This provides a dependence of different symmetries on each other and provides an algebraic structure for the symmetries of a figure. This structure forms the essential tool in the systematic study of symmetries and provides statements about both the necessity and impossibility of certain constellations:

- It is necessary for every 100° -rotationally symmetric figure to be also 200° -rotationally symmetric. On the other hand, a 100° -rotationally symmetric figure cannot exist at all unless it is also 40° -rotationally symmetric, since $4 \cdot 100^\circ = 400^\circ = 40^\circ \pmod{360^\circ}$.
- Any figure with two reflectional symmetries on intersecting lines is automatically rotationally symmetric. However, the converse does not hold, as evidenced, for example, by Figure 6. The figure shown there has several non-trivial rotational symmetries (90° , 180° , 270°), but is not reflectional symmetric.

Questions of this kind can also be part of teaching mathematics in school, even if the term *group* is not explicitly mentioned there as a rule. Furthermore, the group-theoretic view on symmetry for the case of finite symmetry groups also forms the basis for specifying a subset of the symmetry group by which all other symmetries are generated.

4.3 Specifying Symmetry on the Semantic Level – Part 2: Symmetry in School Mathematics

The three *interface aspects* to symmetry support thinking and speaking about phenomena relevant to school mathematics in which symmetry plays a role. To substantiate this, I first discuss the role of the concept of symmetry in school mathematics (S6). Even though I am referring to German curricula (since PSTs, who are the target group of our innovations, are trained for the German school system), the corresponding (international) literature on the identified focal points of the school treatment (e.g., Dreyfus & Eisenberg, 1990; Hoyles & Healy, 1997; Leikin, 2003; Ng & Sinclair, 2015; Son, 2006; Tatar et al., 2014) of the concept of symmetry shows that there are many overlaps with other countries.

A review of the German curricula shows that *symmetry* is a cross-cutting issue that is relevant across all school levels. Thereby, three main foci crystallize (KMK, 2004, 2012; MSW NRW, 2014, 2019):

1. Approaches to different types of symmetries
2. Symmetries as a central property of plane figures
3. Using symmetry for reasoning and problem-solving

I now discuss these in detail. Since this paper is about the specification and structuring of a learning object for a mathematics course, the focus is on the relations to academic mathematics and less on didactical questions about learning symmetry. The latter are relevant for the conception of a subsequent mathematics didactic course but would go beyond the available scope here.

4.3.1 Approaches to Different Types of Symmetries

This section is concerned with locating approaches to the various types of symmetries as they occur in the mathematics classroom from a mathematical point of view.

Obviously, mathematics classes mainly deal with bounded figures. Accordingly, I focus first on the approaches to reflectional and rotational symmetries. In addition, in textbooks, point symmetry (as a special case of rotational symmetry) is often mentioned separately.

For reflectional symmetry, there is a multitude of possible construction and verification activities on different levels of representation. These include, in particular, folding (one half of a cut-out figure can be folded exactly onto the other half) and mirroring with a real mirror (if the mirror is placed at a certain position of the figure, the still visible part of the figure together with the mirror image results in the original figure again).

Already in these examples, it becomes clear that the reflectional symmetry is not approached by an invariance idea (Section 4.2.1), but by the fact that if one mirrors a partial figure at a suitable line, a symmetrical figure is obtained. Thus, constructing a symmetrical figure takes place in the sense of the *aspect of reconstruction and reduction* (Section 4.2.2). This can also be seen in the textbook excerpt shown in Figure 7, where reflectional symmetry is defined by the composition of two “congruent halves”⁶.

Knowledge: Reflectional Symmetry

If you can fold a figure along a straight line in such a way that the two partial figures are congruent, this figure is called **reflectional symmetric**. The straight line is called **axis of symmetry**.

Example 1: Drawing Reflectional Symmetrical Figures

You can also create reflectional-symmetrical figures on paper. Mirror the figure on the red straight line.

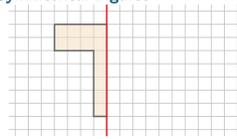


Figure 7. – Translated excerpt from the German textbook *Fundamente der Mathematik* 5, NRW (text and image after Pallack, 2013a, p. 62).

In fact, I could not find any textbooks in which the approach is based on the *aspect of invariance*. Also, various mathematics didactic studies on reflectional symmetry use an (re)constructional approach in the sense of “completion of one half” (e.g., Aktaş & Ünlü, 2017; Götz et al., 2020; Ng & Sinclair, 2015). One possible explanation is that an invariance-addressing enactive verification activity would have to swap orientation. Thus, the figure would have to be turned around in such a way that the axis of symmetry remains invariant. This means a complex three-dimensional activity would be necessary to capture a two-dimensional phenomenon. In German mathematics didactics, this problem of formalizing so-called *elementary kinematic movements* to mathematical functions was discussed for example by Bender (1982).

The situation is different for point symmetry and rotational symmetry: The rotation by an arbitrary angle can be performed as a “two-dimensional activity,” and invariance can be checked, for example, using transparent paper. Accordingly, the approaches found in German textbooks all address the *aspect of invariance* (Section 3.2.2.1). One example can be seen in Figure 8.

6. Cf. footnote 5 in Section 4.2.2 for the problematization of this wording.

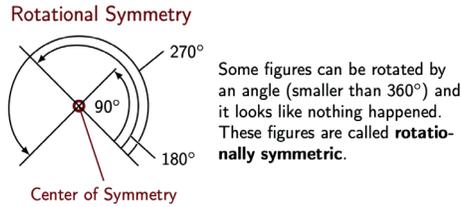


Figure 8. –Translated excerpt from the German textbook *Neue Wege 6, NRW* (text and image after Körner et al., 2013, p. 146).

Also, the *aspect of reconstruction and reduction* (Section 4.2.2) is addressed, however not in the definition but in textbook activities, in which a rotationally symmetrical figure is to be produced by rotating a fundamental domain given.

From a mathematics didactics perspective, these findings are interesting, among other things, because the analyses of textbook definitions using the interface aspects make it clear that the treatment of the two types of symmetry most relevant to school mathematics adopt different mathematical perspectives. As a result, there is a risk that reflectional symmetry and rotational symmetry stand side by side as unrelated concepts without supporting students in developing a general concept of symmetry. Conversely, it is important for the training of PSTs to know these different perspectives and to be able to implement them in teaching practice.

According to Section 3.1.2, the translational symmetries are still missing in this overview. By Theorem 3.6, these can only be treated in the context of unbounded figures. Although such figures play a rather subordinate role in school mathematics, nevertheless also here textbook excerpts can be found (Figure 9).

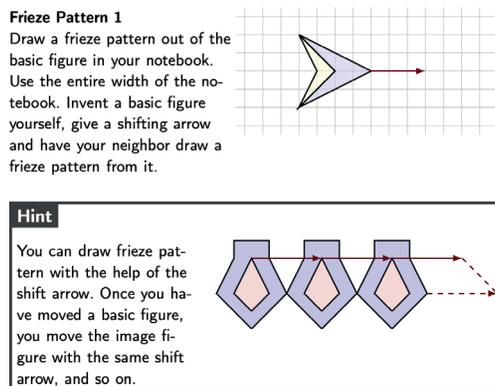


Figure 9. –Translated excerpt from the German textbook *Neue Wege 6, NRW* (text and image after Körner et al., 2013, p. 159).

Strictly speaking, by the unboundedness of the figure, it is not possible to access the invariance enactively. However, if one restricts oneself to a sufficiently large part of the figure, one can still approach the *aspect of invariance* with transparent paper. Focusing on the *aspect of reconstruction and reduction*, the question to be answered is again: Which is an as small as possible building block from which the figure can be constructed via shifting?

Overall, it can be seen that even though in academic mathematics symmetry is usually introduced via an invariance property, in school mathematics the idea of reconstruction also plays a relevant definitional role.

4.3.2 Symmetries as a Central Property of Plane Figures

The investigation of symmetries of plane figures (mostly polygons of different complexity or figures close to everyday life like traffic signs) and vice versa the construction of plane figures with certain symmetries are central classroom activities in the subject area of symmetry. A typical example is the hierarchization of quadrilaterals in the *House of Quadrilaterals* (Franke and Reinhold (2016, p, 128), Weigand et al. (2018, p. 199 f), Figure 10).

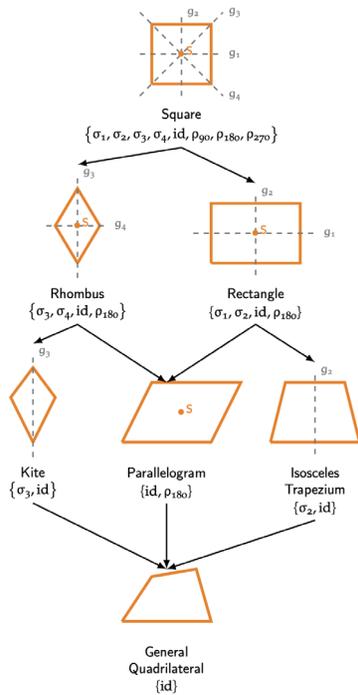


Figure 10. –The House of Quadrilaterals from symmetry perspective. For each quadrilateral, the symmetries are given in the academic mathematical notation of sets via the symmetry group.

Of course, hierarchizing quadrilaterals along symmetry properties is only one of several possibilities. Often, in class, the first sorting is done according to side lengths, parallelism, and/or angle sizes. The examination and analysis of symmetries of the different quadrilaterals provides an occasion in mathematics lessons for local ordering according to the number and type of symmetries. Furthermore, subgroup relations can be studied propaedeutically in a simple example. For making a list of symmetry properties, the *aspect of groups* of the concept of symmetry (Section 4.2.3) provides a helpful mathematical background perspective for the teacher: In the context of the *House of Quadrilaterals* from the reflectional symmetries of rhombus, rectangle and square already necessarily follow the respective rotational symmetries (Theorem 3.5).

Even if the term “group” is not explicitly used in German mathematics classes, one can still find task formats in which the learners work with symmetry in the sense of the *aspect of groups*. For example, in the German textbook *Fundamente der Mathematik 6* (Pallack, 2013b, p. 91), for rotationally symmetric n-gons, all possible angles of rotation are to be described in dependence on n in a systematic way. Another activity is the use of the so-called “mirror book”. This means two mirrors that are fixed to each other along one edge (Figure 11). By changing the angle, different symmetrical figures can be created and examined. This approach addresses the *aspect of reconstruction and reduction* and allows investigations in terms of the *aspect of groups*.

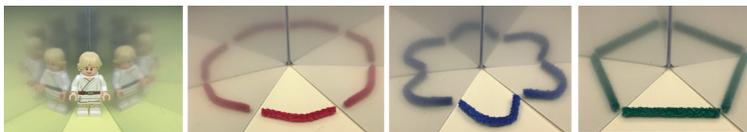


Figure 11. – The “mirror book” can be used to create reflectional symmetric and rotationally symmetric figures. Photos: M. Hoffmann, based on *Mathematik Neue Wege 6*, NRW (Körner et al., 2013, p. 160).

4.3.3 Using Symmetry for Reasoning and Problem Solving

The use of symmetry as a strategy and heuristic principle (e.g., Dreyfus & Eisenberg, 1990) is explicitly listed as an objective in the curricula, at least in Germany. The strength of symmetry here lies mainly in the *aspect of reconstruction and reduction* (Section 4.2.2): If the object to be considered in a problem has symmetric properties, these can be used to reduce the complexity of the object and to generalize statements that hold in one part. On the other hand, a symmetric object can be (re)constructed from an asymmetric one, the consideration of which yields the desired results. The following five examples are typical tasks students must deal with in school mathematics. The examples illustrate both

strategies and, in the sense of (S6), determine a further relevance component related to symmetry as a subject of learning in school mathematics. Other very interesting and instructive examples can be found, e.g., in Dreyfus and Eisenberg (1990, 1998), Leikin et al. (2000), and Leikin (2003).

1. The reflectional symmetry of the isosceles triangle provides the congruence of the two base angles. In the equilateral triangle, both the use of multiple reflectional symmetries and the use of rotational symmetry yield the congruence of all angles.
2. Pythagorean theorem can be justified through constructing a symmetric figure (Figure 12).

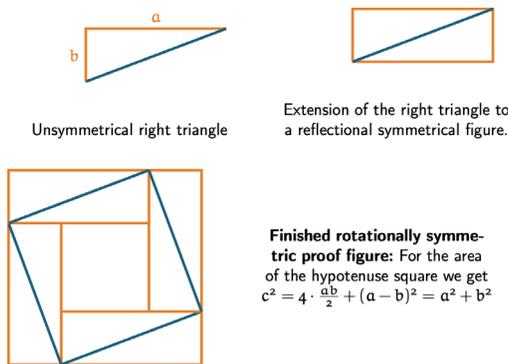


Figure 12. – Proof of Pythagoras’ theorem using a symmetric extension.

3. One can quickly convince oneself of the validity of the theorems of opposite angles and the theorem of alternating angles at intersected parallels using a point symmetry argument.

The first three examples are of an elementary geometric nature and symmetry is used to a certain extent in the classical sense. This is not the case in the following two examples. Here we are dealing with using symmetries in the study of algebraic objects (equations and functions). The symmetry is then used (possibly implicitly) via the function graphs, which can be interpreted as point sets in \mathbb{R}^2 .

4. The solutions of quadratic equations are symmetric about the vertex (the $-\frac{p}{2}$ in the PQ-formula) of the underlying parabola.: The solution(s) of the quadratic equation $x^2 + px + q = 0$ are – if they exist – given by

$$x = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}.$$

5. The knowledge of the symmetries of a function to be investigated allows a significant simplification of work: For example, for functions with point symmetric or reflectional symmetric graphs, only half of the function must be investigated. The translational symmetry of the sine function allows the conclusion from the interval $[0, 2\pi]$ to the entire definition range \mathbb{R} .

The last example is somewhat different, as symmetries here even contribute to the reduction of the dimension.

6. The consideration of complex rotationally symmetric bodies (three-dimensional) can be reduced to the study of a two-dimensional fundamental domain with the standard methods of school analysis. (This example of a fundamental domain differs from the others seen so far in that it is lower-dimensional than the set generated from it, and the associated transformation group is not discrete.)

4.4 Synthesis: Revised and Extended Structure of the topic Symmetry

In this section, a specification of the topic *symmetry* has been done at the semantic level. An essential, constructive result (in connection with the guiding question (S2)) of the specification is the identification of the three *interface aspects* to the symmetry concept: the *aspect of invariance*, the *aspect of reconstruction and reduction*, and the *aspect of groups* (Sections 4.2.1, 4.2.2, 4.2.3).

Subsequently, according to the guiding question (S6), the meaning of the concept of symmetry for mathematics education was described and accordingly (S7) analyzed against the background of the interface aspects from a subject-specific perspective (Section 4.3).

On this basis, the structuring of the learning object can now be revised. The three *interface aspects* specify areas in which horizontal mathematization can take place through appropriate examples (both from the field of academic mathematics and from the school-relevant areas identified in Section 4.3) (S3). The mathematical foundation of the three interface aspects then represents the objective for vertical mathematization (S4).

In connection with the first structuring presented in Section 3.3, it makes sense to first look at school-related examples of the individual types of Euclidean symmetries. Under the guiding question of what these concepts have in common that justifies the common suffix “- symmetry”, the *aspect of invariance* and the *aspect of reconstruction and reduction* can then be dealt with in a first approximation. Based on this, a formalization of the aspects by the definitions of symmetry groups (Definition 3.1), orbits (Definition 3.7), and fundamental domains (Definition 3.8) is possible. Therefore, the advantages of taking the symmetry group (and thus the invariance) as a starting point for theory building must be discussed. Then the group properties (Theorem 3.4) and their consequences for the

dependence of symmetry properties (Theorems 3.5, 3.6) can be focused. After explaining the interface aspects, different areas of the school occurrence of “symmetry” can be classified and analyzed from a mathematical perspective in different methodical settings (e.g., portfolio work). In the end, one can discuss whether there can actually be figures with further symmetry properties and then discuss Example 3.2 and Theorem 3.3, prove them, and classify them on the meta-level. On this basis, further deepening can then take place in different directions.

In relation to the research questions, this section identifies topics at different levels that can be used to supplement the treatment of symmetry at the formal, rigorous level (Section 3) in order to adopt a complementary profession-oriented perspective in the sense of overcoming the second discontinuity.

5 Overview and Outlook on the Concrete and Empirical Level

As already described in Section 2.3, after having dealt in detail with the formal level (Section 3) and the semantic level (Section 4), I give a brief overview of the two remaining levels in this paper.

5.1 The Concrete Level: Core Questions and Contexts for Studying *Symmetry*

The specification and structuring at the concrete level addresses “the realization of the teaching-learning arrangement by core ideas, problems, and situations, in which the mathematical knowledge is relevant and could be constructed in a generic way” (Hußmann & Prediger, 2016, p. 37).

At this level, too, it seems sensible to consistently include a *profession-oriented* perspective in the analysis. This means that, regarding the situations and problems, not only the mathematical content knowledge but also its use as a resource for coping with teaching situations (Section 2.1) must be addressed. It is, therefore, necessary to clarify how contexts and activities may look in which the PSTs can use both mathematical content knowledge (on the formal level) and the results of the analysis on the semantic level (especially the interface aspects) as resources for mastering typical teaching situations in an instructive way.

At this point, we focus on a central innovation that addresses precisely these goals, namely the *interface tasks* already mentioned in section 2.1. The underlying idea of enriching mathematics courses with tasks in which links between school mathematics and academic mathematics are made explicit can be found in many projects in recent years that aim to

overcome the double discontinuity (e.g., Bauer, 2013; Eichler & Isaev, 2022; Suzuka et al., 2009). In the context of the “Geometry for PSTs” course, the notion *interface tasks* specifically refers to tasks in which a typical *teaching situation* is explicitly addressed in addition to a mathematical topic area with the following objectives: “first, to enhance the probability that PST will use their academic CK and skills in future teaching situations and second, to change the attitude that their academic CK and skills are relevant by broadening and deepening PSTs’ views on future typical requirements.” (Hoffmann & Biehler, 2023, p. 739). The tasks are constructed in such a way that, on the one hand, academic mathematics is linked to school mathematics, and on the other hand, the PSTs also use these links to cope with typical teaching situations (in a fictitious context) and to initiate reflection processes on the usefulness of the academic mathematical resources (Hoffmann & Biehler, 2023, p. 742). I illustrate this concept with four examples *interface tasks* on symmetry we used in the course “Geometry for PSTs”. This also shows how the aspects analyzed on the formal and semantic level can be used profitably by teachers. The teaching tasks to which the respective tasks relate are all taken from the list in Prediger (2013, p. 156).

1. The systematization of special quadrilaterals in the *house of quadrilaterals* (Figure 10) is a typical topic in lower-secondary school mathematics. One of our *interface tasks* involves developing the scheme shown in Figure 10, including the formal description by the respective symmetry groups. The PSTs then have to justify on different levels why the classification is complete and why there cannot be any other special quadrilaterals (in terms of symmetry). The group structure and especially the investigation of possible subgroups serve as a mathematical background. This task addresses the teaching task of “mastering requirements for pupils themselves and being able to work on them at different levels”.
2. The same teaching task as in 1. is addressed by an interface task in which the PSTs have to find typical statements from school mathematics that can be justified by a symmetry argument. Possible examples are the contexts in the list at the end of Section 4.3.3.
3. In a further interface task, the PSTs independently develop the findings described in section 4.3.1 on the introduction of different types of symmetry in textbooks. To do this, they analyze relevant excerpts using their knowledge and skills in academic mathematics and, in particular, the interface aspects of symmetry. Thus, they work in the context of “analyzing and evaluating approaches in textbooks.”
4. Another relevant teaching task is to analyze and evaluate students’ contributions and respond to them in a way that supports learning. We also address this in an interface task in which the PSTs deal with the statement of a fictitious student

who claims that there are no other 100° rotationally symmetric figures apart from circles. The PSTs must first clarify this statement from a mathematical perspective and then formulate an answer. In another study, we evaluate the resulting arguments (Hoffmann & Biehler, 2024).

Using example 4, it can be shown how the structure of a profession-oriented treatment of the topic of symmetry, as described in this article, can be the basis for a good PST answer. For the mathematical clarification, for example, Theorem 3.5 (Section 3) can be used to find a 100° rotationally symmetric n -gon like a regular 18-gon. Based on this clarification, an argumentation can be elementarized using the interface aspects presented in Section 4. For example, in a regular 18-gon, the central angle for two adjacent vertices is exactly 20° . So, if one rotates the 18-gon by 20° around the center, each corner lands on its neighbor, and overall, the figure looks exactly the same as before. The regular 18-gon is, therefore, 20-rotationally symmetric (*aspect of invariance*, Section 4.2.1). Using the *aspect of groups* (Section 4.2.3), one can conclude that it has the desired 100° -rotational symmetry. Further details and a detailed analysis of the potential of this task can be found in Hoffmann and Biehler (2024).

5.2 The Empirical Level: What do we know about the learner's perspective?

The empirical level is about “addressing cognitive and possibly social aspects of student thinking, typical resources, pathways, and obstacles” (Hußmann & Prediger, 2016, S. 37). A comprehensive discussion would go beyond the scope of this article. At this point, we will limit ourselves to highlighting the main areas for which relevant studies can contribute to the specification and structuring of the subject of “symmetry”:

- Studies that provide information on the learning level of the concept of symmetry (Weigand et al., 2018, pp. 184-187) that the PSTs are typically at the start of their studies.
- Studies on how adept PSTs/teachers are at using symmetry for problem-solving and reasoning (e.g., Leikin, 2003).
- Studies that generally deal with the problem of the second discontinuity and possible solutions (e.g., Buchholtz et al., 2013; Eichler & Isaev, 2023; Hoffmann & Biehler, 2023).

6 Discussion

This article provides a comprehensive mathematical-didactic analysis of how the highly relevant topic *symmetry* can be prepared for the university education of PSTs. Methodologically, the analysis is embedded in a design research cycle and serves as preparation

for the actual design of learning activities. The procedure of “specifying and structuring mathematical topics” elaborated by Hußmann and Prediger (2016) was adopted for structuring. The focus was on the formal level (Section 3) and the semantic level (Section 4). The concrete and empirical levels were touched on (section 5) but not dealt with comprehensively as their treatment justifies a separate paper.

A particular challenge was that the topic of symmetry was to be considered not only from a mathematical perspective but also from a *profession-oriented* perspective. Behind this is the objective of thinking about the university treatment of a mathematical topic from the outset in terms of using the mathematical knowledge acquired to master typical teaching situations. In this way, the well-known problem of the second discontinuity in university teacher training in mathematics courses is to be counteracted.

Related to the research question RQ1 (What should PSTs learn about the concept of symmetry in a profession-oriented mathematics course?), the essential result consists of a closely interwoven triad of a deductive theory of the concept of symmetry (Section 3), its profession-oriented systematization through the three interface aspects (Section 4.2), and an analysis from a mathematical perspective of the role that symmetry plays in mathematics education (Section 4.3). All three areas can be transferred into course content. Thereby, the newly developed *interface aspects* proved to be very descriptive in analyzing the role of symmetry in the school context (Section 4.3). Other studies conducted in the context of the course *Geometry for PSTs* showed that students, too, consider the interface aspects to be a valuable tool for linking learned academic mathematics with school mathematics (Hoffmann, 2022; Hoffmann & Biehler, 2023). Concerning research question RQ2 (What is a possible structuring of the content identified in answering RQ1?), the structuring presented in Section 4.4, in combination with the contexts presented in Section 5.1, provides an instructive starting point for explicit design activities, such as those presented in Hoffmann (2022).

Regarding the method of analysis, it must be noted that many guiding questions could not be answered in the degree of detail suggested by their formulations. This shows that in the field of university didactics, the multitude of mathematics didactic insights is not yet available, as it is the case regarding mathematics learning in school. I see this as an appeal for more topic-specific research related to university content (not only at the beginning of studies).

From my perspective, the present work shows that for such projects, the chosen methodological approach provides a good orientation and, in particular, through the specific guiding questions, also reveals further topic-specific research gaps.

From a broader perspective, the analytical method presented here aligns with a growing number of international projects in mathematics education for the university training of

PSTs. Although the projects differ in detail, e.g., in whether they take a developmental perspective or a more empirical-measuring perspective on the topic, they have in common the core idea that PSTs need a modified mathematics education that is somehow adapted to their future profession, rather than the education received by students who study mathematics as a major subject. There is a certain consensus that elements of academic mathematical knowledge and skills add value to teaching in the classroom (e.g., Ball & Bass, 2002; Dreher et al., 2018; Hoffmann & Biehler, 2023, 2024; Wasserman et al., 2017; Zazkis & Marmur, 2018; Section 2.2). Regardless of how this “more” or “differently” is formulated in different theories, I think that the analytical approach presented in this paper can be helpful in working out how an adjusted teaching/learning structure can look explicitly on the level of a specific topic. In the analysis of a specific learning object, there is also the potential to address facets of the teacher profession that cannot be theorized as mathematical content knowledge in the classical sense (e.g., Allmendinger et al., 2023, “mathematical orientation”; Prediger & Hefendehl-Hebeker, 2016, “epistemological awareness”; Seaman & Szydlik, 2007, “mathematical sophistication”).

Finally, I would like to mention that the example of the presented considerations clearly shows that the interplay between didactical considerations and mathematical analyses can have great added value. In particular, questions like “Which symmetries can a figure have in a given metric space?” are didactically highly relevant but methodically can only be answered using a formal proof.

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Appendix

Axioms, Definitions, and Theorems

This appendix provides an overview of key axioms, definitions, and theorems used in the paper. The details and proofs can be found in Hoffmann et al. (2024).

Definition A.1 (Metric Spaces): Let X be a set and $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ a function satisfying the following axioms for all $x, y, z \in X$:

- (M1) $d(x, y) = 0 \Leftrightarrow x = y$,
- (M2) $d(x, y) = d(y, x)$,
- (M3) $d(x, y) + d(y, z) \geq d(x, z)$.

Then we call (X, d) a **metric space** and d a **metric**.

Definition A.2 (Metric Straight Line): Let (X, d) be a metric space (Definition A.1). We call a subset $g \subset X$ a **metric straight line** if there exists a mapping $\gamma : \mathbb{R} \rightarrow X$ which has the following properties:

- (1) $\gamma(\mathbb{R}) = g$,
- (2) $\forall s, t \in \mathbb{R} : d(\gamma(t), \gamma(s)) = |t - s|$.

We call such a mapping γ an **isometric parametrization** of the metric straight line g .

Definition A.3 (Isometry): Let (X, d_X) and (Y, d_Y) be metric spaces (Definition A.1). A mapping $\varphi : X \rightarrow Y$ is called **isometry** if for all $P, Q \in X$ holds:

$$d_Y(\varphi(P), \varphi(Q)) = d_X(P, Q).$$

We denote the group of all bijective isometries of X by $\text{Isom}(X)$.

Axiom A.4 (Incidence Axiom): Let (X, d) be a metric space (Definition A.1) with at least two points. We say that X satisfies the **incidence axiom** of plane geometry if for every two points $A, B \in X$ with $A \neq B$ there is exactly one metric line (Definition A.2) which contains A and B .

Axiom A.5 (Reflection Axiom): Let (X, d) be a metric space satisfying the incidence axiom (Axiom A.4). We say that X satisfies the reflection axiom of the plane geometry if for every metric line $g \subset X$ (Definition A.2) holds:

1. (Separation property) The complement $X \setminus g$ of the metric straight line g has exactly two path-connected components. We call these the **sides** of g .
2. (Symmetry property) There is a bijective isometry $\sigma_g : X \rightarrow X$ which fixes g pointwise and interchanges the two sides of g . We call such an isometry a **reflection** on g .

Definition A.6 (Neutral Plane). We call a metric space (X, d) which satisfies both the incidence axiom A.4 and the reflection axiom A.5 a **neutral plane**.

Definition A.7 (Parallelism). Let (X, d) be a metric space. We call two metric straight lines (Definition A.2) $g, h \subset X$ **parallel** (for short: $g \parallel h$), if $g \cap h = \emptyset$ or $g = h$ holds.

Axiom A.8 (Parallel Axiom). Let (X, d) be a metric space. We say that X satisfies the **parallel axiom** if for every metric straight line (Definition A.2) $g \subset X$ and every point $P \in X$ there is exactly one metric straight line $h \subset X$ with $P \in h$ and $g \parallel h$ (Definition A.7).

Definition A.9 (Euclidean Plane). We call a neutral plane (Definition A.6) an **Euclidean plane** if it satisfies the parallel axiom A.8.