Developing Kleinian Praxeologies: The Case of the Integral

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In this paper, we pursue Winsløw's modelling of Klein's second discontinuity, within the Anthropological Theory of the Didactic (ATD), by introducing the notion of Kleinian praxeologies. These new praxeologies are built from praxeological blocks from existing praxeologies, from upper high school and university, to underline their links in mathematics teacher training. Then we present the results of an experiment, conducted according to the methodology of Didactic Engineering, which aims at the development of Kleinian praxeologies by teacher students. Our case study focuses on the integral of upper high school, in its links with Measure Theory taught at the university, in France. These links are described in terms of dominant praxeological models, enriched by Kleinian praxeologies. The data are analyzed using the different tools of the "questioning the world paradigm", in ATD (the questions-answers map and the Herbartian schema, for the study of chronogenesis and mesogenesis, respectively). The methodology allows a fine-grain analysis of the students' work and opens many perspectives for didactic research on Klein's second discontinuity, whether for the study of students' difficulties in linking elementary and advanced knowledge or for didactic engineering that aims to strengthen these links.

Keywords: Klein's second discontinuity, Anthropological Theory of the Didactic, Kleinian praxeologies, measure and integration

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Desarrollo de las praxeologías kleinianas: el caso de la integral

En este trabajo, seguimos la modelización de Winsløw de la segunda discontinuidad de Klein, dentro de la Teoría Antropológica de lo Didáctico (TAD), introduciendo la noción de praxeologías kleinianas. Estas nuevas praxeologías se construyen a partir de bloques praxeológicos de praxeologías existentes, de la escuela secundaria superior y de la universidad, con el fin de subrayar sus vínculos en la formación de profesores de matemáticas. A continuación, presentamos los resultados de un experimento, realizado según la metodología de la Ingeniería Didáctica, que tiene como objetivo el desarrollo de praxeologías kleinianas por parte de los estudiantes de magisterio. Nuestro estudio de caso se centra en la integral del bachillerato, en sus vínculos con la Teoría de la Medida enseñada en la universidad, en Francia. Estos vínculos se describen en términos de modelos praxeológicos dominantes, enriquecidos por las praxeologías kleinianas. Los datos se analizan utilizando las diferentes herramientas del "paradigma del cuestionamiento del mundo", en ATD (el mapa de preguntas-respuestas y el esquema herbartiano en particular, para el estudio de la cronogénesis y la mesogénesis, respectivamente). La metodología permite un análisis fino del trabajo de los alumnos y abre muchas perspectivas para la investigación didáctica sobre la segunda discontinuidad de Klein, ya sea para el estudio de las dificultades de los alumnos para vincular los conocimientos elementales y los avanzados o para la ingeniería didáctica que pretende reforzar estos vínculos.

Palabras-claves: la segunda discontinuidad de Klein, Teoría Antropológica de lo Didáctico, praxeologías kleinianas, medida y integración

Développement des praxéologies de Klein : le cas de l'intégrale

Dans cet article, nous poursuivons la modélisation de la seconde discontinuité de Klein par Winsløw, dans le cadre de la Théorie Anthropologique du Didactique (TAD), en introduisant la notion de praxéologies de Klein. Ces nouvelles praxéologies sont construites à partir de blocs praxéologiques issus de praxéologies existantes, du lycée et de l'université, afin de souligner leurs liens dans la formation des enseignants de mathématiques. Nous présentons ensuite les résultats d'une expérimentation, menée selon la méthodologie de l'Ingénierie Didactique, qui vise le développement de praxéologies de Klein par des étudiants en formation des enseignants. Notre étude de cas porte sur l'intégrale du lycée, dans ses liens avec la théorie de la mesure enseignée à l'université, en France. Ces liens sont décrits en termes de modèles praxéologiques dominants, enrichis par les praxéologies de Klein. Les données sont analysées à l'aide des différents outils du "paradigme du questionnement du monde", en TAD (le question-gramme et le schéma herbartien notamment, pour l'étude de la chronogenèse et de la mésogenèse, respectivement). La méthodologie permet une analyse fine du travail des étudiants et ouvre de nombreuses perspectives pour la recherche didactique sur la seconde discontinuité de Klein, que ce soit pour l'étude des difficultés des étudiants à relier les savoirs élémentaires et avancés ou pour une ingénierie didactique visant à renforcer ces liens.

Mots-Clés : seconde discontinuité de Klein, Théorie Anthropologique du Didactique, praxéologies de Klein, mesure et intégration

Introduction

As early as 1908, Felix Klein identified a problem in the training of school teachers, namely a double discontinuity in the transition from high school to university, and then when student teachers go back to school to teach. To address this, he sets out in a series of books (Klein, 1902-1909/2016) to present "elementary mathematics from a higher perspective" based on three main principles: emphasizing connections between mathematical domains, showing how academic mathematics relates to school mathematics, and connecting mathematics to applications, or intuition to formalism and abstraction (Kilpatrick, 2019). These three principles form the basis of his "plan B" for mathematics teaching.

The research field of mathematics education, in particular the French tradition of "didactics of mathematics" (Artigue, 2019), emerged from this project in the 1960s. Courses designed to recapitulate knowledge from an integrative perspective are now the norm at the end of a study program. In spite of that, the second discontinuity seems to remain (Wasserman, 2018) and calls for more research on the transfer of academic knowledge into teacher-relevant knowledge. While one might naively believe or wish for this, there is no reason why such transfer should be automatic. Few studies have provided evidence, but we can point to the recent empirical results of Hoth et al. (2020) on secondary school teacher education in Germany.

What kind of mathematical knowledge is useful to a future teacher? What types of links are developed and are to be developed between university knowledge and school mathematics, in the training programs, to promote professional development of teachers? What elements of didactics of mathematics should be used to supplement this knowledge and how should these elements be taught? These questions which are currently debated make Klein's second discontinuity a lively topic in mathematics education research and a strong issue for the profession of mathematics teacher and teacher-trainer.

In this direction, new tools have recently been brought forward by Winsløw and Grønbæk (2014) and Winsløw (2020) who posed the issue raised by Klein within the theoretical framework of the Anthropological Theory of the Didactic (ATD) (Chevallard & Bosch, 2020). Winsløw uses the notion of *relation* of an individual to an object of knowledge within an institution. He distinguishes between high school (*HS*) and university (*U*), as well as three different *institutional positions*: student in high school (*s*), student in university (σ) and finally teacher in high school (*t*). An object of knowledge (in our case, the integral), which lives through both institutions, will be denoted *o* in *HS* and ω when it is a theory of integration (Riemann or Lebesgue, in connection with Measure Theory) taught at *U*. Winsløw and Grønbæk (2014) then propose the following model of Klein's discontinuities:

$$R_{HS}(s,o) \to R_U(\sigma,\omega) \to R_U^*(\sigma,\omega) \to R_U(\sigma,o) \to R_{HS}(t,o)$$

where Klein's answer to the problem of transfer consists in establishing a relation $R_U^*(\sigma,\omega)$ weaving together o and ω in view of the change of position that the last arrow expresses. In a later model, Winsløw (2020) denotes by $R_U(\sigma, o \cup \omega)$ this new integrating relationship and introduces new symbols to designate the didactic knowledge and know-how useful for the construction of $R_{HS}(t,o)$.

In a previous work (Planchon & Hausberger, 2020), we designed a problem of the type proposed for the written exam of the CAPES[1] and dealing with the high school integral in its links with the Riemann integral and Measure Theory. As stated in the Official Journal[2], "The notions dealt with in these programs [from high school] must be able to be approached with a hindsight corresponding to the first year of the master's cycle". A relation of the type $R_U(\sigma, o \cup \omega)$ is thus expected from a future teacher in France. The design of the problem was inspired by the methodology implemented by Winsløw & Kondratieva (2018): the links are described through relations between praxis and logos blocks (see next section) related to the mathematical knowledge at stake in *HS* and *U*. Moreover, the study of historical epistemology has made it possible to identify in the work of Lebesgue (1975) an axiomatics, more elementary than that of Measure Theory, which is conducive to founding the notion of area. Its adaptation by Perrin (2005), while reinforcing the role of geometric transformations in the spirit of modern algebra, confirms these potentialities. The problem was submitted to a class of students preparing for the CAPES: the results of this experiment showed that most students succeeded in grasping the axiomatics in the first questions of the problem, but that they lost sight of its function when it came to engaging in a proof of the fundamental theorem of analysis by refraining from reading the properties of areas on the figure. This confirms the need for learning devices that aim to develop links between mathematical knowledge from *HS* and *U*.

In this article, we present the results and the methodology of a second implementation of the epistemological ideas underlying the CAPES problem, but in a completely different didactic modality: a study and research activity (SRA, next section). We hypothesize that promoting questioning, through an SRA, will foster the confrontation of *U* and *HS* knowledge. From the point of view of theoretical contributions, we pursue the formalization in

1. Certificat d'Aptitude au Professorat de l'Enseignement du Second degré. This is the main competitive examination for the recruitment of teachers in France, which evaluates the mastery of disciplinary knowledge covering essentially the first two years of university, as well as abilities linked to professional dimensions.

2. of December 8, 2015, text 8; this text describes the contents and modalities of the recruitment competition.

ATD of Klein's ideas by introducing the notion of Kleinian praxeology. Our main research question may thus be formulated in the following way: "How to foster the development of Kleinian praxeologies by students, in the case of the integral?" The global methodology of the study follows that of didactic engineering (Artigue, 2020). This research question, which includes both a theoretical and an experimental component, will be refined into more concrete sub-questions in the course of the article.

The article begins with a brief presentation of ATD and the multiple standard SRA tools that are used in our analyses. We appeal in advance to the eagerness of the non-ATD specialist to learn about ATD from the vast literature available, whenever needed. Although we emphasize the core ideas, the constrained format of the article does not allow us to offer a more detailed account of these notions, which are nowadays increasingly used in university mathematics education research (Hausberger & Bosch, 2022; Nardi & Winsløw, 2018). We defend the view that a theoretical framework is the pillar of the scientificity of a scientific field in its ability to describe phenomena beyond naive points of view. This is why we wish to offer the reader analyses that take advantage of the full strength of the ATD framework, without amputating notions that proponents of strict theoretical economy would deem secondary in a tight argument about the results produced. Our view is that this would result in a loss of nuance and coherence, which is of course open to debate, but is not the subject of this article.

The notion of Kleinian praxeology and the overall methodology are the subject of the next section. We then present our praxeological models (fourth section), which constitute the epistemological reference of the knowledge taught in *HS* and *U*, on the integral. The empirical study, at the heart of our contribution, is the subject of the fifth section: we present the experimental device, then provide a priori and a posteriori analyses, which are compared to discuss the relevance of the device. We conclude with the results of the study and the significance of our methodologies and theoretical tools for the study of Klein's second discontinuity.

Theoretical framework

This study is anchored in the Anthropological Theory of the Didactic (ATD), which intervenes in our study under three major aspects. Firstly, it provides the language to model Klein's double discontinuity. Secondly, the theory of praxeologies play a key role in the design to connect *HS* and *U* knowledge. Finally, the tools of the "questioning of the world paradigm" (Chevallard, 2012) will allow for a fine-grained analysis of the students' work.

Relationship to objects

As mentioned in the introduction, ATD puts forward the relativity of objects of knowledge *o* to the institutions *I* that develop, normalize, and transmit them, as well as the subjection of persons *x* to these institutions (Chevallard & Bosch, 2020). Thus, ATD will not be interested so much in the persons as in the generic positions *p* they occupy, notably that of teacher *t* or student *s.* It is thus a question of studying the *institutional relations* $R_I(p,o)$ of individuals of the institution *I*, in position *p*, to the object of knowledge *o*.

The arrow diagram from Winsløw and Grønbæk (2014) reproduced in the introduction condenses the different institutions, institutional positions, and knowledge objects involved in Klein's double discontinuity. From our point of view, it is a strength of ATD to offer symbolic writings that allow such a synthesis to be made, in line with the strength of symbolism in mathematics in general.

In our study, we are mainly concerned with institutions where mathematics is taught. However, it is important to keep in mind that this knowledge is the result of a historical process, carried out within the scholarly sphere, and then of a process of didactic transposition, which modifies certain aspects under the effect of institutional conditions and constraints.

Praxeologies, praxeological models and ostensives

Praxeologies lie at the foundations of ATD, which places the analysis of human activities at the heart of its endeavor. A praxeology *P* is the union of a praxis Π and a logos Λ, the discourse on the praxis. ATD postulates that the relationship $R_I(p,o)$ emerges from the praxeologies where the object *o* intervenes, at the different possible levels of the praxeology: the type of tasks *T*, the technique τ used to solve the tasks *t* of this type, the technology θ which justifies τ, or the theory Θ which offers praxis its ultimate foundation.

This set of praxeologies can be described in the form of a structured model that is called a *reference praxeological model* (RPM; Florensa et al., 2015). RPMs are reconstructions of the knowledge to be taught, obtained by considering different levels of the didactic transposition (via historical epistemology, official programs, textbooks, and teaching materials). We will describe RPMs by highlighting the unification of punctual praxeologies around common technologies, to constitute *local* mathematical organizations (LMOs), themselves unified by the theoretical level within *regional* organizations (RMOs). The RMOs correspond in general to sectors and the LMOs to themes that structure the presentation of the course.

In fact, the RPM can have different functions, the two main ones being either to describe the knowledge as it is taught in a given institution, in which case we speak of a *dominant* *praxeological model,* or to serve as an anchor point for didactic engineering, which can lead the researcher to enrich or transform a dominant model according to the didactic issues at stake.

The set of praxeologies known to a person constitutes her *praxeological equipment*. In fact, a learner may be unable to mobilize a praxeology in a given situation, even though this praxeology is part of the equipment. To analyze these phenomena, Bosch and Chevallard (1999) have introduced the notion of *ostensives* to designate the signs (sound, visual, material) to which our senses give us access, engaged in mathematical activity. Ostensives refer to other signs or concepts (non-ostensives), what Bosch and Chevallard call the *semiotic valence* of an ostensive. Ostensives thus carry the semiotic point of view in ATD. An ostensive can evoke praxeologies, and thus open up a potential for action by the subject, which Bosch and Chevallard call the *instrumental valence* of the ostensive. The absence of certain key ostensives or *triggering ostensives* (*ostensifs déclencheurs*; Aray-Chacón & Matheron, 2015) such as the symbol *P* activating the sector of probabilities during the calculation of a Gaussian integral (Hausberger et al., 2021), is an explanatory factor of non-activation of expected praxeologies. A didactic gesture of the teacher consists then in pointing out the missing ostensive. Another factor is the loss of instrumentality linked to a deficient praxis or logos, if the technique is poorly assimilated or the technology insufficient to allow the application of the technique in a partly unusual context (Hausberger et al., 2021).

Didactic organizations and moments of the study

Once learning goals have been described in praxeological terms, the question arises as to how to teach these didactic stakes, i.e., the associated *didactic organization*. It is at this level that the *moments of the study* (Chevallard, 2002a) intervene, as many episodes which have the function of allowing the development of a given praxeology.

According to ATD, a study process comprises the moment of the *first encounter* with the praxeology where the type of tasks is identified, the moment of the *exploration* of the type of tasks and the *emergence of a technique*, the *technological-theoretical moment* where the validity of the technique is discussed, the moment of *working of the technique* where one tends to improve and routinize the technique, then the moments of *institutionalization* and *evaluation*. The first three moments constitute Group I.

The exploratory moment and the moment of emergence of the technique are strongly weakened when the pedagogical organization is based on lectures. This tends to render the *raison d'être* of mathematical concepts invisible, since they do not emerge from the need to enrich techniques, during problem solving.

Paradigm of questioning the world and its tools

It is to remedy this and to strengthen questioning that ATD has introduced a new approach, which is called the *paradigm of questioning the world*, in opposition to the *monumentalist paradigm* (Chevallard, 2012). This approach is based on the notion of *study and research path* (SRP) (Bosch, 2018), where the study of a generating question Q will lead a collective X of learners, aided by a group Y of study aids, in other words a didactic system $S(X;Y;Q)$, to produce a *milieu* M from which will finally emerge an answer $R^{\heartsuit}$ to the question Q.

Again, using symbols to synthesize ideas, such a study process is noted by its Herbartian schema:

$$[S(X;Y;Q) \Rightarrow M] \Rightarrow R^{\heartsuit}$$

The milieu M is constituted by the official answers $R^{\diamond}_{i}$, which are deconstructed by the didactic system by visiting works W_i, (which will be for us praxeologies P_j or blocks of praxeologies Π_j or Λ_j, or even constituents of these blocks), by producing new questions Q_k.

It is classical in ATD to study the *chronogenesis*, i.e. the temporality of the study process, through the questions and answers that appear. This model, called *questions-answers map*, can be represented in the form of a mathematical diagram (Bosch, 2018). The fruit of the *mesogenesis* (the genesis of the milieu) is given by the developed Herbartian schema, i.e. detailing M:

$$M = \{Q, Q_1, \ldots, Q_k, R^{\diamond}_{1}, \ldots, R^{\diamond}_{n}, W_1, \ldots, W_p\}$$

The last axis of analysis, *topogenesis*, consists in describing the roles of the various actors. In our case, the set Y is essentially reduced to the teacher, who leads the study process.

In fact, an SRP is intended to take place over a long period of time. In our case, the experimentation took place over two three-hour sessions. We therefore speak of a *study and research activity* (SRA) rather than SRP. Nevertheless, an SRA can be analyzed with the same tools as an SRP.

Research questions and methodology

Kleinian praxeologies and research questions

Winsløw's model of Klein's double discontinuity highlights the need to create a new relationship $R_U(\sigma, o \cup \omega)$ in view of bridging o and ω. *Kleinian praxeologies* aim at fulfilling this need. They emerge from the consideration of dominant models for *HS* and *U* and will be of two types.

The first, denoted P^*, is based on a praxeology P encountered at U, of which certain components have been modified. The goal is to generate $R_U(\sigma, o \cup \omega)$ from above, starting from university knowledge.

The second type aims at linking P^* praxeologies to effective secondary education practices, thus considering $R_U(\sigma, o \cup \omega)$ from the bottom. A Kleinian praxeology of type $P^\sim$ is thus based on a high school praxeology P the logos and/or praxis of which is enriched in contact with P^*.

It goes without saying that the definitions of Kleinian praxeologies given here remain very abstract and general at this stage. As in mathematics, the definitions will become concrete through the examples we give in the rest of the article.

By adding Kleinian praxeologies to the dominant praxeological models, we obtain our reference praxeological model for the didactic engineering.

Our main research question can be stated synthetically as follows: how to foster the development of Kleinian praxeologies by the student, in the case of the integral? This is already a refined form of the general problem of implementing Klein's ideas, which stems from theoretical ideas. New sub-questions are added: first, what are typical Kleinian praxeologies that emerge in the case of the integral? What didactic organization should be put in place? In the language of ATD, it is a question of adapting the moments of the study to Kleinian praxeologies. Then, from a methodological point of view, how to analyze the students' work? Do they develop the intended Kleinian praxeologies? If not what hinders the development of Kleinian praxeologies?

Methodology

Our work is based on the methodology of didactic engineering (Artigue, 2020). The latter classically divides the researcher's work into four phases: preliminary analyses; design of the experiment and a priori analyses; experimentation and data collection; a posteriori analysis and validation of the device. This validation is internal, in the sense that it is carried out by comparing a priori and a posteriori analyses, i.e. by discussing whether or not the device produces the prescribed effects. Describing our methodology in more detail consists in specifying how we conducted each of the four phases.

Our preliminary analyses include an epistemological analysis, based on the work of historians. However, this is done essentially through the identification of the role of the work of Jordan and Lebesgue, who have nourished the design (as explained in the introduction). We will limit ourselves, in the context of this article, to mentioning this aspect of the epistemological analysis. The elaboration of dominant praxeological models (DPM; next section), is also part of the preliminary analyses. For this purpose, we used official programs, textbooks, and course documents (handouts and tutorial sheets).

The design of the experiment amounts to transforming the CAPES problem into an *SRA guide* (the document submitted to students) and to specify its didactic organization. The problem (Planchon & Hausberger, 2020) was constructed based on the DPMs. However, the notion of Kleinian praxeology has not been identified at this stage of the research. It is a praxeological analysis of the CAPES problem, with a focus on the links between the two models (*HS* and *U*) thus created, that has allowed them to be highlighted, first in the context of the problem and then in the more general form that we have just presented. The examination of key tasks of the problem that lead to those Kleinian praxeologies, applying a principle of economy of didactic time, allowed the elaboration of the SRA guide.

As we planned to develop an SRA, we used the corresponding tools to describe the study processes generated by these tasks. For our study, therefore, we have considered that the DPMs describe the institutional praxeological equipment. A gap may exist between this equipment and the praxeological equipment of a student. But the DPMs were used to determine the a priori milieu (thus the Herbartian schema). From the reflexive examination of the a priori questions-answers maps came a formalization of the moments of the study, for Kleinian praxeologies.

The choice of data collected was dictated by the goal to draw the a posteriori questions-answers maps and Herbarian schemas. We will detail these data in the fifth section. The examination of the milieu is the key element to attest the development of Kleinian praxeologies, while the a posteriori questions-answers maps, linked to the moments of the study, inform us about a possible blockage at the level of a given stage of the study process. Finally, the examination of the ostensives present or absent in the milieu, in connection with the notions of semiotic and operative valence of ostensives, allows us to shed light on the reasons for these possible blockages, partly in the form of hypotheses.

Dominant praxeological models (DPM)

We present the main elements of the DPMs corresponding to the integral taught in French high schools and to Measure Theory as it is taught at the University of Montpellier, limiting ourselves to what is crucial for the engineering.

High school DPM

The integral is taught at the *Terminale* level in France (last year of upper high school). The notion of integral of a continuous positive function is defined as the area under the curve, the area remaining an intuitive notion referring to a praxis introduced in elementary school: the counting of unit squares contained in the considered region of the plane, after partitioning the region in squares, and the procedures of surface cutting by elementary polygons (rectangles, triangles). Of course, this procedure does not apply to any curve, so that the existence and definition of the area under the curve, in general, remain based on sensitive intuition. There is no definition based on the limits of areas of rectangles in the official high school curriculum. Such Riemann sums are calculated in a few rare exercises.

This definition of the integral (a technology denoted Θ_{area}) generates three punctual praxeologies (we name the praxeologies by highlighting the type of tasks): calculation of an integral using an area formula, calculation of the area between two curves and approximation of the integral by the method of rectangles (the technique consists in dividing the considered interval in a finite number of sub-intervals $\left[x_i;x_{i+1}\right]$ and in approximating the integral as the sum of the areas of the rectangles of height $f\left(x_i\right)$ or $f\left(x_{i+1}\right)$ obtained from this subdivision). The praxeological development is accompanied by an enrichment of the theoretical block: the growth property of the area (if $A \subset B$ then $\mu(A) \leq \mu(B)$, where μ designates the area as a function applied on surfaces) and the invariance by symmetry (if s is an isometry of the plane, then $\mu\left(s(A)\right)=\mu(A)$) are new elements of the logos which appear as this first LMO develops. Indeed, the growth property of the area justifies the bounding of the area under the curve by the sums of the areas of the rectangles under and above the curve obtained from subdivisions, and the invariance by isometry allows to justify that the area over the curve of a negative function *f* is equal to the area under the curve of -*f* (hence the definition of the integral of a negative function).

The fundamental theorem of calculus (FTC, the differentiability of the area function) provides, as a corollary, the new tool for the calculation of integrals of positive functions via antiderivatives. Its proof is mentioned in the "demonstrations" section of the official program, that points the proofs to be worked on with students. We denote by t_{HS} the isolated task "prove the FTC for continuous and monotone functions".

Remarkably, a change of perspective on the integral occurs, which is defined in the syllabus, in the case of a continuous function of any sign, via an antiderivative: the integral between a and b of f is $F(b)$-$F(a)$, where F is an arbitrary antiderivative of f. The existence of such a function F is a consequence of the FTC: indeed, a continuous function f on a closed interval admits a lower bound m; the function f-m is positive and therefore admits a primitive function G according to the FTC, so that the function $F(x)=G(x)+mx$ is an antiderivative of f. The FTC thus becomes a theoretical element related to three new praxeologies (calculation of an integral via a primitive function, integration by parts, finding lower and upper bounds for an integral) which constitute a second LMO unified by the new antiderivative-based perspective on the integral. Finally, the logos Λ_{HS} is enriched with new properties of the integral (linearity and growth).

University DPM

We have analyzed different materials (syllabus, tutorial sheets, solutions of exercises and lecture notes taken by a student) of the course “measure and integration” of the third year of the Bachelor at the University of Montpellier, in 2020. The Riemann integral is taught in the second year, but it does not provide further mathematical insights on the notion of area than the Riemann sums. The main goal of the course is to depart from geometry and give a rigorous foundation of the high school integral and the FTC based on the real numbers and the formal concept of limit. In particular, we have not modeled any type of task that explicitly mentions areas. Riemann sums are employed for the calculation of limits of sequences without geometric interpretation. By contrast, measure theory can be seen as an axiomatic and general modeling of the processes of measurement of geometric quantities. This is the viewpoint we followed in the didactic engineering, which is why our DPM at the university level is focused on this course.

The course outline suggests a division into four sectors in the sense of ATD: measure theory, general integration theory, pushforward and product measures, and L^p spaces. We will detail the praxeologies for the first sector only since the sector of the general theory of integration (Fubini theorems, integration with respect to a measure, Lebesgue convergence theorems), and the next sectors are beyond the scope of this article. Indeed, our experiment is centered on the notion of area and the FTC.

The notions of σ-algebra and measure are concepts of measure theory as a structuralist theory. The concrete/abstract and particular/general dialectics, which Hausberger (2017) has subsumed under the name dialectics of objects and structures, thus apply. The structuralist method proceeds by reasoning in terms of classes of objects, stability of properties of structures by operations on these structures. We can assume that such structuralist principles guide the presentation of the course as well as the types of tasks proposed.

The notion of σ-algebra on a set, that is the class on which measures are applied, is axiomatically defined[3] in the vocabulary of set theory. This notion unifies a first LMO generated by two types of tasks: prove that a set is a σ-algebra and prove that a σ-algebra is generated by a subset. The second theme is “measurable sets and functions”. The stability of the σ-algebra structure by direct and inverse image under a function is studied. New technological elements appear according to the structure of the codomain of the studied functions (vector space, ring, topology), which allows to establish stability properties of measurable functions with respect to arithmetic operations on functions and taking a limit.

3. A σ-algebra $\mathcal{A}$ on a set X is a set of parts of X such that $\mathcal{A}$ is nonempty, closed under complements and under countable unions.

Finally, the third theme is centered on the axiomatic definition[4] of a measure. This definition generates a punctual praxeology (prove that a given application is a measure) which is illustrated on particular cases: the Dirac measure, the counting measure, the Lebesgue measure on $\mathbb{R}^n$. The tutorial sheets allow us to identify two others punctual praxeologies within the LMO. First, $P_{M,1}$ (prove a property of a specified measure) appears through many instances, e.g. the task $t_{M,1}$ (prove that a measure on $\mathbb{R}$ invariant by translation is diffuse[5]). The technique consists in mobilizing the properties of the considered measure, by a direct reasoning or by the contradiction. The technology θ_M contains the general properties of measures (axiomatic definition, growth, countable subadditivity, increasing and decreasing limits) and the theory Θ_M the proofs of these properties which mobilize set theory. The technique is grounded on the application of the axiomatic method: to take advantage, as much as possible, of the generalizing and simplifying point of view offered by structures.

Then, the praxeology $P_{M,2}$ (measuring a set for a given measure) is developed: its technique consists in identifying how the set is constructed with respect to operations that characterize σ-algebras, and then to use the properties of the measures. An example of assignment of this type of tasks is illustrated in **Figure 1** (question 2).

Exercice 11. For any interval $I \subset \mathbb{R}$ and any $a \in \mathbb{R}$, we denote $I = \{x + a \mid x \in I\}$.
Let μ be a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that :

- $\mu([0; 1[) = 1$;
- For any interval $I \subset \mathbb{R}$, and any $a \in \mathbb{R}$, we have $\mu(I + a) = \mu(I)$.

The goal is to prove that μ is the Lebesgue measure.

(1) Show that $\mu(x) = 0$ for any $x \in \mathbb{R}$. *This is called diffuse measure.*
(2) Show that $\mu([0; x[) = x$ for any $x \in \mathbb{R}_+^*$. *You can start by proving it for every rational $x \in \mathbb{Q}_+^*$.*
(3) Deduce that $\mu = \lambda$ (the Lebesgue measure).

Figure 1. – Excerpt from a tutorial sheet on Measure Theory at the University of Montpellier (in French)

Note that the emphasis is put on a property characteristic of the Lebesgue measure on $\mathbb{R}$: the invariance by translation. The technique consists, after having treated the case of x integer and rational, in using the density property of $\mathbb{Q}$ in $\mathbb{R}$ and in writing $[0,x[$ as the increasing union of a sequence of intervals $[0,q_n[$ and then in using a property of measures,

4. A positive measure on $(X,\mathcal{A})$, where $\mathcal{A}$ is a σ-algebra on X, is an application μ defined on $\mathcal{A}$ with values in $\mathbb{R}^{+}$, such that $\mu(\emptyset)=0$ and, for any countable family A_i of pairwise disjoint elements of $\mathcal{A}$, $\mu(\cup A_i)=\Sigma\mu(A_i)$ (i.e. σ-additivity).

5. A measure μ is on $\mathbb{R}$ is *diffuse* if for all x in $\mathbb{R}$, $\mu(\{x\})=0$.

linked to σ-additivity: the limit of the measure of an increasing sequence of measurable sets is the measure of the countable union.

To conclude, the sector dedicated to measure theory may be modeled by an RMO consisting of three LMOs around, respectively, the notion of a σ-algebra, the definition of measurable functions, and the definition of a measure.

Empirical study

We first present the experimental setup (its institutional context and the SRA guide we developed), then we provide a priori and a posteriori analysis of the SRA, which are compared to discuss the relevance of the device in a final section.

Presentation of the experimental device

Institutional context

The experiment took place in a teaching unit of the Master MEEF[6] at the University of Montpellier. During their first year, MEEF students prepare for both the CAPES in mathematics (which essentially evaluates mastery of the discipline) and entry into the teaching profession (professional skills). Most of them have a bachelor's degree in mathematics from the University of Montpellier. We therefore assume that their praxeological equipment contains what is described in the university DPM.

The experiment took place in the context of the Covid-19 pandemic, in April 2021. At that time, students were confined to their homes and attended distance learning classes. The experiment was part of a teaching unit on epistemology and didactics centered on high school programs and organized around main mathematical domains (algebra, analysis, probability, etc.). The first author was in charge of the analysis theme. The whole class (sixteen students) participated in the experiment, which lasted two three-hour sessions. In this article, we present and analyze only the first session, our goal being to share the methodology used and the results that such device produces.

The students had at their disposal a dedicated worksheet (the SRA guide) that will be described below. They were divided into four groups of four students and the collaborative work, in synchronous distance mode, was organized using different tools. First, the BigBlueButton platform was used to create virtual rooms: a common room, reserved for the institutionalization and devolution phases, as well as four private rooms for work in

6. Métier de l'Enseignement, de l'Education et de la Formation.

small groups. Second, the googleDoc platform was used: each group had a shared document and was instructed to keep tracks of their work, including informal notes.

At the end of this experiment, we collected several data: the recordings of the audio exchanges of the students in the private rooms, those of the institutionalization phases in the common room, and the shared files of the different groups.

Presentation of the SRA guide

Riemann integration is, of course, relevant for this experimentation. As we indicate in a note on p. 22, the second part of the activity involves elements of praxeologies related to the Riemann integral. The choice to begin with the axiomatic perspective of measure theory seems to us to better correspond to the construction that is taught in French high school, namely, to rely on an informal axiomatic approach to areas. The SRA guide therefore begins with the introduction of a notion new to students, called "area measure", by its axiomatic definition: An area measure is an application $\mu : \mathcal{D} \to \mathbb{R}_+$ that satisfies the following axioms:

- $\forall A, B \in \mathcal{D}, A \cap B = \varnothing \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$ (additivity)
- If $A \in \mathcal{D}$ and s is an isometry, then $\mu(s(A)) = \mu(A)$ (invariance under isometry)
- If C is the semi-open unit square, $C = [0;1[\times[0;1[$, then $\mu(C) = 1$

Here $\mathcal{D}$ denotes a nonempty collection of subsets of $\mathbb{R}^2$ that is closed by finite union and finite[7] intersection and that contains all polygons, points and segments. This axiomatics corresponds to a modern version of Jordan and Lebesgue's measure of so-called "quarrable sets" (Lebesgue, 1975) that was identified through our epistemological investigation. The effective construction of the measure and of the set $\mathcal{D}$ is the focus of another classroom activity (another SRA), which won't be discussed in this paper.

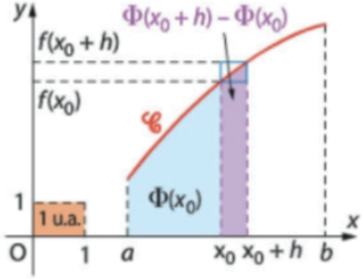
The SRA guide highlights three tasks and a textbook excerpt (**Figure 2**). The relevance of these tasks for the development of Kleinian praxeologies will be discussed in the context of the a priori analysis. The three tasks assigned are :

- t_1: show that the area measure is diffuse (for each x in $\mathbb{R}^2$, $\mu(x) = 0$)
- t_2: determine the area measure of a rectangle according to its dimensions and justify your answer;
- t_3: based on the area measure, rewrite the proof of the FTC presented in the textbook with the university standard of rigor.

7. If A and B are in $\mathcal{D}$, then $A \cup B$ and $A \cap B$ are in $\mathcal{D}$. Countable union stability is not necessary here (unlike in general measure theory) and $\mathcal{D}$ is not a σ-algebra.

Theorem : Let f be a continuous and positive function on an interval $[a;b]$. Then the function $\phi : x \mapsto \int_a^x f(t)\mathrm{d}t$ is differentiable on $[a;b]$ and $\phi' = f$.

Proof : We consider the case where f is increasing on $I = [a;b]$. x_0 and h are two numbers : $x_0 \in I,\ h \neq 0$ and $x_0 + h \in I$.



- If $h > 0$, since f is increasing, $f(x_0) \leqslant f(x_0 + h)$. $\phi(x_0 + y) - \phi(x_0)$ represents the area under the curve $\mathcal{C}$ over $[x_0; x_0 + h]$. We bound this area by the areas of two rectangles with the same width and heights $f(x_0)$ and $f(x_0 + h)$:
$$h \times f(x_0) \leqslant \phi(x_0 + h) - \phi(x_0) \leqslant h \times f(x_0 + h).$$
Hence, $f(x_0) \leqslant \frac{\phi(x_0+h)-\phi(x_0)}{h} \leqslant f(x_0 + h)$.
- If $h < 0$, we can similarly show that : $f(x_0 + h) \leqslant \frac{\phi(x_0+h)-\phi(x_0)}{h} \leqslant f(x_0)$.
- Now, as f is continuous at x_0, we have $\lim_{h \to 0} f(x_0 + h) = f(x_0)$. Therefore, by the squeeze theorem, $\lim_{h \to 0} \dfrac{\phi(x_0 + h) - \phi(x_0)}{h} = f(x_0)$. Thus, ϕ is differentiable at x_0 and $\phi'(x_0) = f(x_0)$.

This result holds for all x_0 in $[a;b]$, so ϕ is differentiable on $[a;b]$ and $\phi' = f$.

Figure 2. – Proof of the FTC
(translated from textbook "Transmath", (Bonneval et al., 2012))

Moreover, the SRA guide begins with a meta-discourse that problematizes the three tasks, including an excerpt from the epistemologist Blanché (2009, p.55) to shed light on the role of axiomatics in contemporary mathematical activity:

> Le but qu'on se propose quand on met sous forme axiomatique une théorie déductive, c'est de la dégager des significations concrètes et intuitives sur lesquelles elle a d'abord été construite, afin d'en faire clairement apparaître le schéma logique abstrait.[8]

Based on this quote, the learning goal is stated in the SRA guide as follows: "We wish, by this work, to mobilize your mathematical knowledge (in particular Measure Theory) in

8. The purpose of putting a deductive theory into axiomatic form is to free it from the concrete and intuitive meanings on which it was first built, in order to make its abstract logical scheme clearly appear.

order to base the intuitive notion of area on an axiomatic adapted to secondary education". In other words, the generating question of our SRA may be expressed in the terms of Q_θ: "*how to give an axiomatic foundation to the intuitive notion of areas on which the high school integral is based*?" Students were given 30 minutes per task to work in the private rooms.

A priori analysis

We present for each task the a priori questions-answers map describing the chronogenesis as well as the Herbartian schema synthetizing the mesogenesis. We describe the questions that, in our view, should emerge through the study, as well as the answers that the study group can provide, along with the works visited to provide these answers. The works visited are linked to the DPMs presented in previous section whose notations are used. The a priori analysis of each task leads to the description of the Kleinian praxeology whose development is aimed at. Finally, we present our adaptation of the study moments for Kleinian praxeologies.

Analysis of the first task

The task t_1 poses at once the question of the technique allowing its realization, that is to say Q_{t_1}: "*How to prove that the area measure is diffuse*?". The ostensives "measure" and "diffuse" give rise to a reactivation of the praxeology $P_{M,1}$ (prove a property of a specified measure), in particular its instantiation $t_{M,1}$ (prove that a measure on $\mathbb{R}$ invariant by translation is diffuse): the question of the application of the technique $\tau_{M,1}$ in this new context is thus raised. The work $\Pi_{M,1}$ (praxis of the praxeology $P_{M,1}$) is visited, from which $Q_{t_1,1}$ is raised: "*What are the general properties of measures which are still valid in the context of the area measure*?" Two sub-questions $Q_{t_1,2}$ "*What are the general properties of measures*?" and $Q_{t_1,3}$ "*How are these general properties proved* ?" emerge to identify respectively the general properties of measures and the main ingredients to prove these properties. Thus, the work Λ_M (definition and general properties of measures) is visited: the answer $R^\lozenge{}_{1,2.}$ "*σ-additivity, growth, countable subadditivity, increasing and decreasing limits, invariance by translation (for the Lebesgue measure)*", which coincides with the work θ_M, and the answer $R^\lozenge{}_{1,3}$ "*Some properties use the notion of σ-additivity, others the simple additivity*" is brought by the visit of the work Θ_M (axiomatic of measure).

The study of the proofs thus allows to discriminate the properties of measures that apply in the context of the area: invariance by translation and growth are preserved. This constitutes the answer $R_{1,1}$ to question $Q_{t_1,1}$ and allows the construction of a new logos $\Lambda^*{}_M$, related to a new praxeology $P^*{}_{M,1}$ whose praxis is similar to that of $P_{M,1}$. This praxeology of type P^* can be described as follows:

$T^*_{M,1}$: Prove a property of the area measure

$\tau^*_{M,1}$: Mobilize the properties of the area measure, by direct or indirect reasoning

θ^*_M : Properties of the area measure (invariance by translation, growth)

Θ^*_M : axiomatics of the area measure

The Herbartian schema for the first question is therefore:

$$[S(X;Y;Q_1) \Rightarrow M_1] \Rightarrow R_1^{\heartsuit} = P^*_{M,1}$$

where $M_1 = \{Q_1, Q_{1,1}, Q_{1,2}, Q_{1,3}, R^{\diamond}_{1,2}, R^{\diamond}_{1,3}, \Pi_{M,1}, \Lambda_M, \Lambda^*_M\}$.

The technique $\tau^*_{M,1}$ is in fact split in two possible techniques, $\tau^*_{M,1,1}$ and $\tau^*_{M,1,2}$, adapted from $P_{M,1}$. We explain them in **Figure 3** for the reader who wishes to go into the details of the mathematical technique, by applying them to the context of the task t_1. Note that, in both cases, the growth property of the measure must be validated by the students: the area measure does not satisfy the axioms of a measure in the sense of Measure Theory, so that proofs from Measure Theory need to be adapted to the new context.

$\tau^*_{M,1,1}$. Let us suppose that $\mu(\{O\}) = \varepsilon > 0$. Since $\mathbb{R}$ is archimedean, there exists a natural number n such that $n\varepsilon > 1$. Consider the set of points $\{A_k, k \in \{0, \ldots, n-1\}\}$ where for any $0 \leq k \leq n-1$, $A_k = (\frac{k}{n}, 0)$. Since the measure μ is invariant by translation, we have :

$$\forall k \in \{0, \ldots, n-1\},\ \mu(\{A_k\}) = \mu(\{0\}) = \varepsilon.$$

But then, as $\bigcup_{k=0}^{n} A_k \subset C$ (the unit square), we deduce by growth of the measure and by simple additivity that $\mu(C) \geqslant n\varepsilon > 1$ which is impossible.

$\tau^*_{M,1,2}$. Let M be a point of the plane; by invariance of the measure by isometry, hence by translation (of vector $\overrightarrow{OM}$), it is sufficient to show that $\mu(\{O\}) = 0$. For all $n \neq 0$, let $C_n = [0, \frac{1}{n}[\times [0, \frac{1}{n}[$ be the square of side $1/n$. For any non-zero natural number n, $\mu(\{(0,0)\}) \leqslant \frac{1}{n^2}\mu(C_1)$, and therefore, taking the limit, $\mu(\{(0,0)\}) = 0$.

Figure 3. – Techniques adapted from $P_{M,1}$

Analysis of the second task

The task t_2 explicitly poses Q_2: "*what is the area measure of a rectangle as a function of its dimensions?*". It is an instantiation of $T_{M,2}$ (measuring a set for a given measure) which was encountered by students in the context of measure theory. Thus, the praxis $\Pi_{M,2}$ is *the* first work visited. Consistent with the realization of $t_{M,2}$, the new sub-questions arising in the milieu consist of proving the formula when the dimensions are, consecutively, integers ($Q_{2,1}$: "*How to measure a rectangle with integer sides*"), rationals ($Q_{2,2}$), positive real numbers x and y ($Q_{2,3}$).

After applying partitioning procedures which mobilize the additivity for the cases of integers and rationals, the work W_{R} (density of $\mathbb{Q}$ in $\mathbb{R}$) is visited to move on to the real numbers. We consider a rectangle of dimensions x and y (two strictly positive real numbers), parallel to the axes. The density allows to justify the existence of sequences of rational numbers (x_n) and (X_n) (resp. $(y_n),(Y_n)$) which converge to x (resp. y), with $x_n \leq x \leq X_n$ and $y_n \leq y \leq Y_n$. Geometrically, the rectangles $[0,x_n]\times[0,y_n]$ and $[0,X_n]\times[0,Y_n]$ bound the rectangle $[0,x]\times[0,y]$. To conclude, the work θ^*_M is visited again, especially the invariance by isometry and the growth of the area measure[9], which are crucial to justify the passage to the limit. Finally, any rectangle is isometric to a rectangle of the type $[0,x]\times[0,y]$. The answer $R_2^{\heartsuit}$ is obtained: "*a rectangle with side lengths real numbers x and y has an area measure xy*".

The Herbartian schema for the second question is therefore:

$$[S(X;Y;Q_2)\Rightarrow M_2]\Rightarrow R_2^{\heartsuit}$$

where $M_2 = \{Q_2, Q_{2,1}, Q_{2,2}, Q_{2,3}, R_{2,1}, R_{2,2}, R_{2,3}, \Pi_{M,2}, \theta^*_M, W_{\mathbf{R}}\}$.

The answer $R_2^{\heartsuit}$ enriches the technology θ^*_M. The targeted praxeology $P^*_{M,2}$, again of type P^*, can be modeled by:

$T^*_{M,2}$: Measuring a set with the area measure

$\tau^*_{M,2}$: Decompose the set into elementary sets

θ^*_M : Properties of the area measure

Θ^*_M : Axiomatics of the area measure

Analysis of the third task

The last task deals with a rewriting the FTC proof (task t_{HS}), as taken from a textbook, but in the university standard of rigor. The aim is to mobilize the logos Λ^*_M. The assigned question is Q_3 : "*How to make the proof of FTC rigorous by relying on the axiomatics of areas?*".

A first sub-question, induced by the SRA guide (that mentions properties possibly read on the figure before quoting Blanché), is $Q_{3,1}$: "*Where does the intuitive notion of area come into play in the textbook proof?*" The ostensive "area" appears explicitly in the proof. Moreover, the ostensive "integral" refers to the area under the curve through the work θ_{area} (the definition of the integral of a continuous positive function as the area under the curve), hence the answer to the question $Q_{3,1}$. The following answer to Q_3 can then be given:

9. Countable additivity is not needed here; the use of the sandwich theorem is enough to conclude.

Let us consider the set $\Omega_t = \{(x,y) \in R^{,2}, 0 \le x \le t, 0 \le y \le f(x)\}$. We have to assume[10] that Ω_t is in $\mathcal{D}$. We can write $\phi(x_0+h)-\phi(x_0)=\mu(\Omega_{x_0+h} \setminus \Omega_{x_0})$ by additivity of the measure. The growth of *f* on the interval $[x_0, x_{0+h}]$ justifies the bounding (in the sense of inclusion) of $\Omega_{x_0+h} \setminus \Omega_{x_0}$ by two rectangles of width *h*, and the growth of the area measure μ, as well as the measure of rectangles, finally give the expected lower and upper bounds for $\phi(x_0+h)-\phi(x_0)$. The definition of the derivative function at a point and a calculation of limits then allow to conclude.

To do so, a formalization work is needed: geometrical objects at stake are designated and linked to subsets of $\mathbb{R}^2$. Properties of the area measure useful to complete the proof are then identified by visiting Λ_M^*, to produce $R_3^\heartsuit$.

The Herbartian schema for this last question is thus:

$$[S(X;Y;Q_3) \rightarrow M_3] \rightarrow R_3^\heartsuit$$

where $M_3 = \{Q_3, Q_{3,1}, R_{3,1}, \Lambda_M^*, \theta_{area}, \Lambda_{HS,M}^{\sim}\}$.

The targeted Kleininian praxeology $P_{HS,M}^{\sim}$, of type $P^{\sim}$ and of which t_3 is an instantiation, may be stated as follows:

$T_{HS,M}^{\sim}$: Write the proof of a proposition that mobilizes the intuitive notion of area with the university rigor standard

$\tau_{HS,M}^{\sim}$: Designate the geometrical objects at play, link them to subsets of $\mathbb{R}^2$, and use properties of the area measure to complete the proof

$\theta_{HS,M}^{\sim}$: Properties of the area measure (invariance by translation, growth, measure of rectangles); axiomatics to separate the logical and the intuitive

$\Theta_{HS,M}^{\sim}$: Area axiomatics; paramathematical notions of rigor and proof at university

It should be noted that the logos comprises a *paramathematical*[11] component, related to the notion of rigor: indeed, this logos is carried by Blanché's quote and we expect students to explicit conceptions on rigor when solving the task.

10. A second activity (Planchon, 2022) introduces an explicit definition of the set $\mathcal{D}$, and students have to prove that the set Ω_t is in $\mathcal{D}$. Continuity of *f* is fundamental (although growth of *f* is sufficient). This mobilizes elements of praxeologies linked to Riemann theory, leading to the development of new Kleinian praxeologies.

11. These notions or elements of discourse on mathematics are useful to the mathematical activity but are not defined mathematically.

Moments of the study of Kleinian praxeologies

Let us now focus on the didactic organization, which amounts to describing the different moments of the study of Kleinian praxeologies.

The moment of the first encounter of P^* praxeologies is carried out with the logos Λ^* (and not the type of tasks): indeed, didactic time does not allow to reconstruct the area axiomatics. On the other hand, the targeted Kleinian P^* praxeologies are based on existing praxeologies. This leads us to modify Chevallard's group I of the usual moments of the study (see theoretical framework) as follows:

- Moment of the first encounter with the praxeology, via the logos Λ_M^*
- Identification of a type of tasks T_M within the DPM related to ω at university
- Implementing a corresponding technique τ_M
- Confrontation of the logos Λ_M and Λ_M^*
- Elaboration of τ_M^* and validation by θ_M^*

Let us explain this new model in the case of question Q_1. When studying the praxeology $P_{M,1}^*$, the study consisted in identifying the type of task $T_{M,1}$, and then implementing the technique $\tau_{M,1}$ ($\Pi_{M,1}=[T_{M,1}, \tau_{M,1}]$), in the framework of measure theory. The discrimination of the properties available (or not) to transpose $\tau_{M,1}$ in the context of the area measure constitutes the moment of confrontation of the logos Λ_M and Λ_M^*. Finally, the last moment corresponds to the implementation of the technique τ_M^* and its validation by θ_M^*. This model applies in the same way to Q_2.

In the case of $P_{HS,M}^{\sim}$, the moment of identification of the type of tasks is not experienced, since it is replaced by the moment of the study of the product of a technique τ_{HS} from high school (here, the proof of the FTC). The confrontation of the logos Λ_{HS} (integration in high school) and $\Lambda_{HS,M}^{\sim}$ (the union of Λ_M^* and a paramathematical component on rigor) leads to the implementation of $\tau_{HS,M}^{\sim}$. To summarize, the moments of the study of the Kleininan $P^{\sim}$ praxeology are:

- Moment of the first encounter with the praxeology, via the logos $\Lambda_{HS,M}^{\sim}$
- Study of the product of the implementation of a high school technique that mobilizes the notion of area
- Confrontation of the logos Λ_{HS} and $\Lambda_{HS,M}^{\sim}$
- Elaboration of $\tau_{HS,M}^{\sim}$ and validation by $\theta_{HS,M}^{\sim}$

A posteriori analysis

We analyzed with the same tools (praxeologies, Herbartian schema and moments of the study) the empirical data collected during the experimentation. From a transcript of students oral discussions in each group, we identified the main questions raised as well as the answers provided by the members of the group. Sometimes, we synthesized their sentences to underline the core arguments. Furthermore, to identify the works that were visited (blocks of praxeologies already developed), we looked in these dialogues, as well as in the written solution that the group collectively produced, for key ostensives related to the praxeologies of Measure Theory described in the University DPM. In our modelling, the milieu constructed by each group therefore consists of the main questions, answers and works that have emerged during the study process. We then compare this milieu with the one constructed in the a priori analysis. We present here some excerpts, translated into English, of the students' dialogues and written solutions as warrants for our claims and modelling in terms of the Herbartian schema.

A posteriori analysis of the first task

Only group A (comprised students A_1 , A_2 , A_3 and A_4) was able to produce a technique for the realization of t_1. The type of tasks $T_{M,1}$ was first identified and the technique $\tau_{M,1}$ seemed to be available. Indeed, the following dialogue points to $T_{M,1}$ and its instantiation $t_{M,1}$ (in the context of the Lebesgue measure, although a different measure is considered):

> A_2: We must show that the Lebesgue measure is diffuse
>
> A_1: We did this exercise in L3 with Lebesgue, do you remember this exercise?
>
> A_2: We have to do it by the absurd. We'll suppose that the measure of a point is epsilon.
>
> A_3: I'm wondering if we shouldn't do with convergences.

The appearance of the word 'Lebesgue' is, for us, an indication that a university praxeology is being employed. This indicates that the second moment of the study is experienced. The last two interventions show the attempt to implement $\tau_{M,1}$: A_2 evokes a proof by contradiction while A_3 evokes the direct proof. We also model the last intervention of A_3 above as $Q^A_{1,1}$ "*Can we use the notion of convergence?*", where the group is indicated by a superscript letter.

Later in the exchanges, we note:

> A_2: In fact no, with what is said in the introduction, what is given on the measure should be enough.
>
> A_3: Ok, this is the basic thing. We have the measure of the union which is the sum of the measures.

For us, this dialogue corresponds to the moment of confrontation of the logos Λ_M and Λ^*_M, then the rejection of the direct proof technique, which uses the notion of limit (and thus $R^A_{1,1}$). This leads the group to the implementation of $\tau_{M,1}$ and we note $Q^A_{1,2}$ "*Can we implement the technique by contradiction* $\tau_{M,1,1}$?". As we have seen in the a priori analysis, the answer $R^A_{1,2}$ requires the confrontation of Λ_M and Λ^*_M. The validation of the technique is ensured by the growth of μ, which is not verified here. A technique is then produced by the group, as reflected in the shared document (**Figure 4**). The growth of μ, which allows to show the final contradiction, is not evoked. Thus, the moment of validation of the technique does not seem to be experienced.

Let $\varepsilon > 0$. $\mu(1,1) = \varepsilon$, so $\mu(1,1) + \mu(3/2,3/2) = 2\varepsilon$ because $\mu(s(3/2,3/2)) = \mu(3/2,3/2) = \mu(1,1)$.

Therefore, $\mu(3/2,3/2) = \varepsilon$. By this reasoning, all points have a measure of ε.

Now, our figure C has an infinite number of points. Let's denote $n \in \mathbb{N}$ such that n is greater than $1/\varepsilon$.

Then, for all $i \in [0;1[$, $\mu\left(\bigcup_{i=1}^{n}\{(i,i)\}\right) = \sum_{i=1}^{n}\mu(\{(i,i)\})$, which is greater than 1 but contained within the square C.

However, the square has a measure of 1. Therefore, there is a contradiction, leading to $\mu(\{(1,1)\}) = 0$.

Figure 4. – Answer to Q_4 from group *A*

To summarise, the questions raised by Group A are thus $Q_1, Q^A_{1,1}, Q^A_{1,2}$ and the corresponding answers are noted $R^A_{1,1}, R^A_{1,2}$. The visited works are the blocks of the praxeology $P_{M,1}$ as well as the new logos Λ^*_M. We can then provide the developed Herbartian schema that accounts for the students' study process:

$$[S(X_A;Y;Q_4) \rightarrow M_1^A] \rightarrow R_{1,A}^{\heartsuit}$$

where $M^A_1 = \{Q_1, Q^A_{1,1}, Q^A_{1,2}, R^A_{1,1}, R^A_{1,2}, T_{M,1}, \tau_{M,1}, \Lambda_M, \Lambda^*_M\}$.

The groups *B* and *C* also identified the type $T_{M,1}$ as indicated by their exchanges, for instance:

> B_3: We have to show that the Lebesgue measure is diffuse?
>
> B_1: No, not really. It is for μ which is the area measure. It is not the Lebesgue measure.

The milieu produced by group *B* therefore contains $Q^B_{1,1}$ "*Should we show that the Lebesgue measure is diffuse*?" and its answer $R^B_{1,1}$, but the group does not engage in an attempt to implement $\tau_{M,1}$.

This lack in the praxeological equipment of the collective is also evident in the work of group *C*:

C_1 : Do you remember how we used to show that the Lebesgue measure is diffuse?

C_4: No idea.

The question raised by the group concerns the implementation of $\tau_{M,1}$ and the absence of answer in the milieu indicates the non-availability of this technique. Finally, group D didn't move further than the moment of first encounter since the task type $T_{M,1}$ was not identified.

A posteriori analysis of the second task

During the study process, the technique $\tau_{M,2}$ (see fourth section on the DPM) was implemented by all four groups for the particular case of integers. We now detail the mesogenesis related to the group whose milieu produced was the richest.

D_1: If they are integers, it's easy. Because we can decompose with small unit squares.

D_2: Here, we will find length times width.

D_1: Nevertheless, two small squares side by side, there will remain the common intersection, it is the side of the square.

D_2: Yes, but since we admit that the measure of the segment is zero.

D_1: Yes, good point. Yes, so it doesn't change anything. So we just have to count the squares.

This dialogue indicates a visit of Λ_M^* with the property of the area measure of segments and the justification of the technique by θ_M^*.

After producing $Q_{2,1}^D$ "Can we decompose a rectangle with integer sides?" and its answer $R_{2,1}^D$ "we can decompose with small unit squares", the collective studies the case of rationals.

D_2: At first, it would be necessary to deal with the rationals.

D_1: Yes. We can say that the idea is the same, a rational can be written p/q, so we can find an integer by which to multiply it to get an integer.

We notice an attempt to implement the technique, but rather in one dimension. The answer to $Q_{2,2}^D$ "*what to do with the rationals*?" is not explicit and the group did not explicitly address the rational case.

Finally, the group tries to study the case of the real numbers by mobilizing the notion of limit, which indicates the visit of the work W_{R} (density of $\mathbb{Q}$ in $\mathbb{R}$):

D_1: We should see if we have 2 irrationals. Can we reduce it to a decomposition? ($Q_{2,3}^D$)

D_2: Then, if we do with the rationals, can we use the limit? ($Q_{2,4}^D$)

D_2: If we take an irrational, there will always be a sequence of rationals which will converge.

Here, we see again a partial implementation of $\tau_{M,2}$, which mobilizes the notion of limit. But the moment of confrontation of Λ_M and Λ^*_M, which would have favored the substitution of the notion of limit by growth properties of μ, has not been experienced. The answer R_2^D may be read in the shared document (**Figure 5**).

For irrational numbers, can we use the fact that for all irrational numbers x, there exists a sequence of rational numbers that converges to x? If ℓ and L are irrational numbers, there exist two sequences (r_n) and (r'_n) that converge respectively to ℓ and L. The rectangle with dimensions r and r' has a measure of $\mu(R)$. The area of rectangle R tends towards the area of the rectangle with dimensions $\ell \times L$. So? Is the area of the rectangle with dimensions $\ell \times L$ the limit of the area of rectangle R.

Figure 5. – Answer to Q_2 from group *D*

With the notations introduced above, the Herbartian schema for this group is thus the following:

$$[S(X_D;Y;Q_2) \Rightarrow M_2^D] \Rightarrow R_{2,D}^{\heartsuit}$$

where

$$M_2^D = \{Q_2, Q_{2,1}^D, Q_{2,2}^D, Q_{2,3}^D, Q_{2,4}^D, R_{2,1}^D, \Pi_{M,2}, \Lambda^*_M, W_R\}.$$

Here, we can see that questions $Q_{2,2}^D$, $Q_{2,3}^D$ and $Q_{2,4}^D$ did not receive a response, as $R_{2,2}^D$, $R_{2,3}^D$ and $R_{2,4}^D$ are not present in the milieu constructed by this group. The notion of density was evoked by another group, but the collective did not succeed in implementing the whole technique $\tau^*_{M,2}$ for the case of real numbers (this group stopped at the bounding of the lengths of the two sides by sequences of rationals). The two other groups stopped at the case of rational numbers.

A posteriori analysis of the third task

The questioning around the notion of rigor appeared in each of the groups, which is attested by the following remarks (from different groups): "*We must have everything justified by our axioms*", "*We must restrict to using* μ", "*We must put the proof at our level*". The confrontation of the logos Λ_{HS} and $\tilde{\Lambda}_{HS,M}$ was thus experienced by the different collectives.

We now detail the study of group *C*. The moment of confrontation of the logos Λ_{HS} and $\tilde{\Lambda}_{HS,M}$ produces $Q_{3,1}^C$ "*How to formulate with* μ?", which expresses the goal to identify the role played by the intuitive notion of area in the proof, and to mobilize the axiomatics:

> C_3: The areas of the rectangles, you put them in μ, and the area of the middle, we leave in $\phi\phi$. No, the middle thing too, should be put in μ?

> C_1: Well yes, it's the measure of the thing you're looking for, so you have to put it with μ.
>
> C_4: Yes, you are right. We note A the subspace, the part between x_0 and $x_0 + h$, under the curve and above the *x*-line. I think we should explain why we can bound with μ.

We notice here gestures of formalization, e.g. designating by A a geometrical object, still read on the figure. There is no sign of transfer from the domain of geometry to the domain of numbers by way of set theory. This final proposition produced by C_4 constitutes, for us, the response to $Q^C_{3,1}$. On the other hand, we can model the discussion below with the question $Q^D_{3,2}$ "*How to justify the bounding* $\mu(R) \leq \mu(A) \leq \mu(R')$?" which arises and generates the visit of the work Λ^*_M:

> C_3: This is what we saw just before, in question 2. The area of the rectangles.
>
> C_4: Why is there the smallest or equal?
>
> C_1: Is it enough to justify that the rectangles are smaller, and therefore the area is smaller?
>
> C_4: Ah but no, it is because μ it is increasing! It is necessary to say that they are included [one set in the other].

This dialogue illustrates the moment of validation of the technique by means of the technology $\theta_{\widetilde{HS,M}}$. The technique $\tau_{\widetilde{HS,M}}$ is implemented by groupe C in the shared document as follows (**Figure 6**).

> Let's denote A as the region enclosed by the curve, the x-axis, and the lines with equations $x = x_0$ and $x = x_0 + h$. Consider R as a rectangle with length $f(x_0)$ and width h.
>
> Similarly, let R' be a rectangle with length $f(x_0 + h)$ and width h.
>
> $R \subset R'$. We have $R \subset A \subset R'$, and due to the growth of μ, we can write :
>
> $$\mu(R) \leq \mu(A) \leq \mu(R')$$
>
> This leads to :
>
> $$f(x_0)h \leq \mu(A) \leq hf(x_0 + h)$$
>
> Now, dividing both sides by h (because $h \neq 0$) :
>
> $$f(x_0) \leq \mu(A)/h \leq f(x_0 + h)$$

Figure 6. – Answer to Q_3 from group C

The milieu constructed by the group C thus contains the questions $Q_3, Q^C_{3,1}, Q^C_{3,2}$ and their answers $R^C_{3,1}$ and $R^C_{3,2}$ ("*it's due to the growth of* μ") as well as the visited works $\Lambda^*_M, \theta_{area}, \Lambda_{\widetilde{HS,M}}$. With the above notations, the Herbartian schema for this group is:

$$[S(X_C; Y; Q_3) \Rightarrow M_3^C] \Rightarrow R_{3,C}^{\heartsuit}$$

where $M^C_3 = \{Q_3, Q^C_{3,1}, Q^C_{3,2}, R^C_{3,1}, R^C_{3,2}, \Lambda^*_M, \theta_{area}, \Lambda_{\widetilde{HS,M}}\}$.

Although the confrontation of Λ_{HS} and $\Lambda_{\widetilde{HS,M}}$ was experienced by the other groups, these collectives failed in elaborating $\tau_{\widetilde{HS,M}}$. Indeed, one of the groups tried to reconstruct the proof of the theorem within the framework of Riemann's integration theory seen in the second year of the bachelor and missed the issues of conversion between the geometrical and the numerical registers, since Riemann's framework is immediately set in set theory. Another group remained blocked at the stage of this conversion: unlike group *C*, it did not designate the geometric objects by letters and was not able to continue the reasoning based on the figure. The last group remained at the stage of explaining the need to mobilize axioms and identified growth as a key argument, but the formalization work was not engaged.

Summary of results and discussion

We conclude with a summary of the main findings, to answer the remaining research questions: do students develop Kleinian praxeologies? If not, what are the obstacles to this construction?

The comparison of the Herbartian schema of group A with that presented in the a priori analysis shows that the generating question Q_1 did indeed generate the expected milieu: works from the DPM related to measure theory were visited. The analysis in terms of the moments of the study also indicates that, for this group, the different moments of the construction of a Kleinian P^* praxeology were experienced: identification of the type of tasks and implementation of an existing technique from university, confrontation of the logos, then adaptation of the technique. The moment of validation of the technique remains weakly experienced; at this point in the study, the function of axiomatics is not yet fully perceived and the growth property remains implicit. The instrumental valence of the ostensive "diffuse" appears to be a key point in the study of Q_1: group A thus produced $\tau^*_{M,1}$, whereas two other groups were blocked by the loss of instrumentality of this ostensive. Indeed, although the type of tasks was identified, the moment of the study corresponding to the implementation of $\tau_{M,1}$ was not experienced, which blocks the elaboration of $\tau^*_{M,1}$.

In the study generated by Q_2, the expected technique was implemented by all four groups for integers. In the milieu constructed by group D, we noticed the validation of the technique by a technological element previously developed: the area measure of segments is zero. This allows the extension of additivity to sets whose intersection is of measure zero, which indicates a step back from the intuitive notion of area and an appropriation of the axiomatics.

The semiotic valence of the ostensive "domain of numbers" is quite high since this ostensive referred to other signs (the articulation between $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$) in three groups. Nevertheless, even if the density of $\mathbb{Q}$ in $\mathbb{R}$ is evoked (the work W_{R} is present in the milieus of two groups), it is not operative: the link between density and the order relation in $\mathbb{R}$, which leads to the implementation of bounding techniques, is not observed in the productions of these students. The moment of confrontation of Λ_M and Λ^*_M, which would have led to highlighting the growth of the area measure, was not experienced by any group. In the end, the technique $\tau_{M,2}$ from measure theory was not implemented. We can hypothesize that this weakness in the praxeological equipment of the students is linked to the conceptual difficulties of the notion of density specified above. This hinders the development of the Kleinian praxeology aimed at, around the technique $\tau^*_{M,2}$.

Finally, during the study process generated by Q_3, the questioning around the notion of rigor appeared in each of the groups. Thus, the moment of the confrontation of Λ_{HS} Λ_{HS} and $\Lambda^{\sim}_{HS,M}$ was experienced by the different collectives, and the elements θ_{area} and $\Lambda^{\sim}_{HS,M}$ are present in the milieus produced. This confrontation of the logos allowed the different groups to grasp the need for formalization, but they were not able to implement it, except for group C. Indeed, this group designated the geometrical objects in play by symbols to mobilize the measure μ. The formalization is not completely achieved, because the designated objects do not appear as subsets of $\mathbb{R}^2$ so that some properties are still read on the figure. Nevertheless, we can see, in this group, some formalization gestures. This group is the only one to have proposed a rewriting of the proof of the textbook. The conversion from the geometric to the number register requires expressing the geometric objects in terms of subsets of $\mathbb{R}^2$, which engages set theory and thus goes beyond the changes of registers experienced in high school. This is a key moment in the completion of the task. Our results suggest that this conversion remains a barrier for students teachers, which hinders the development of the intended $P^{\sim}$ Kleinian praxeology. The prior construction of the P^* praxeologies on which the $P^{\sim}$ praxeology is based thus proved insufficient for the construction of the latter. A weak point in the praxeological equipment of future teachers lies in the praxeologies of formalization and modeling. We can make the hypothesis that they are insufficiently developed in the current curricula.

General conclusion and perspectives

In this paper, we have proposed an experiment to study Klein's second discontinuity through a device realizing Klein's plan B, for the high school integral o in its links with Measure Theory ω taught at university. The formalization in terms of praxeologies has led us to enrich the dominant praxeological models of high school and university with praxeologies of a new type: Kleinian praxeologies, which are of two types, P^* and $P^{\sim}$. These are adapted from university and high school praxeologies respectively, with P^* praxeologies preparing the way for the development of $P^{\sim}$ praxeologies, particularly as the logos of the latter praxeologies is derived from that of the former (Figure 7). Precisely, we have proposed an adaptation of the moments of the study to the case of Kleinian praxeologies, to specify the didactic organization of our device.

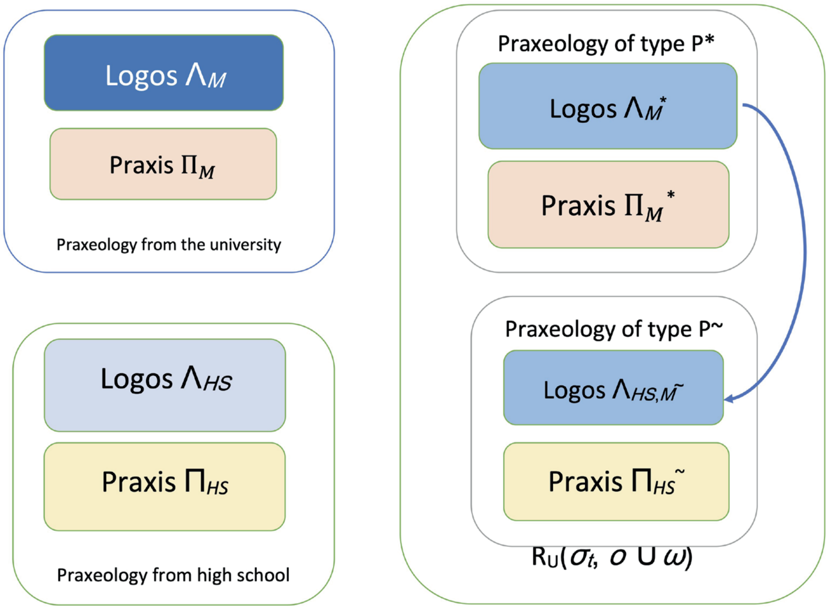


Figure 7. – Kleinian praxeologies

To promote questioning, we chose a study and research activity modality for this device. The analysis of the students' work was thus carried out with the different tools developed in the framework of research on the paradigm of questioning the world, in ATD. This contributes to the originality of our research, as such an approach is new in the literature on Klein's double discontinuity in ATD. Moreover, the a posteriori analysis confirmed the potentiality of our device for the development of Kleinian praxeologies: during each sub-SRA, the main moments of the study were experienced by study collectives. The finesse of the analysis tools also made it possible to situate the obstacles to the development of Kleinian praxeologies in relation to these moments of the study, and then to link the difficulties observed to losses of instrumentality of certain key ostensives and to weaknesses in the students' praxeological equipment. These weaknesses concern praxeologies of measure theory, praxeologies of analysis mobilizing the order relation in $\mathbb{R}$ and praxeologies of formalization-modeling, based on set theory. Our results thus suggest that a certain evolution of the curriculum would be beneficial, even necessary to the development of Kleinian praxeologies. One can also think of making better use of the Measure Theory course materials in the possession of students, in the spirit of the media-milieu dialectic (Chevallard, 2008). In fact, the online mode did not allow the teacher to play his classical role of media as much as it would have been desirable, and this weakness of the device was not anticipated.

This work opens many perspectives, such as the formalization of Kleinian praxeologies with other objects of study than the integral and the experimentation of study and research activities based on these formalizations. One can think of the study of certain aspects of elementary arithmetic in connection with the notion of ideal in Ring Theory, of geometry based on linear algebra and elements of Group Theory, etc. This formalization may be achieved by relying on existing literature that report pedagogical innovations targeting Klein's second discontinuity. The multiplication of examples will provide an empirical basis from which it will be possible to refine the theoretical description of Kleinian praxeologies (for example, in the form of a typology) and their associated moments of study. It will also be an opportunity to test the robustness of our proposed methodologies for task design and data analysis. Finally, $P^{\sim}$ praxeologies could also be enriched by considering the teacher's didactic praxeologies P, such as the evaluation of the validity of answers given by students, or the construction of examples verifying certain properties for classroom activities, etc. This requires crossing our research with work on the teachers' professional gestures, which will be expressed in the language of praxeologies to link them with our praxeological models.

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