Early elementary school children’s use of tables while solving Vergnaud’s additive problems

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This paper describes early elementary school children’s responses to additive problems when using paper and pencil representations, specifically when using a table. We explore the following research question: *In what ways do tables influence young students’ (Grades 1-3; ages 6-8) accuracy and ability to work with additive problems including identifying unknowns and components of the problems?* Children from a public school in a diverse suburb of the Northeast of the United States were interviewed individually. Each child was presented with six additive problems taken from Vergnaud’s (1982) work. We designed three representational contexts (plain paper and pencil, unlabeled tables or labeled tables), to which children were randomly assigned. We highlight three findings from this study. First, our data emphasize that what children can do depends on the problem context and the tools available to them. Second, our data illustrate how some representations help children with problems that involve a composition of two transformations, but not necessarily with problems of transformation between two measures. Moreover, children are able to respond to some problems more successfully when they are able to engage with them through the use of specific representations. Third, some representations facilitate an explicit attention to types of quantities and problem structure. Implications for instruction are also discussed.

Keywords: elementary mathematics, additive problems, representations

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El uso de tablas entre niños en los primeros grados de escolaridad mientras resuelven problemas aditivos de Vergnaud

Este artículo describe las respuestas de niños de escuela primaria a problemas aditivos cuando usan representaciones con papel y lápiz, específicamente cuando usan una tabla. Exploramos la siguiente pregunta de investigación: *¿De qué manera influyen las tablas en la precisión y la capacidad de los niños pequeños (primer a tercer grado de primaria; 6 a 8 años) para trabajar con problemas aditivos, incluida la identificación de incógnitas y componentes de los problemas?* Se entrevistó individualmente a niños de una escuela pública en un suburbio diverso del noreste de los Estados Unidos. A cada niño se le presentaron seis problemas aditivos tomados del trabajo de Vergnaud (1982). Diseñamos tres contextos representacionales (hoa en blanco, tablas sin etiquetas y tablas con etiquetas), a los que se asignaron niños de forma aleatoria. Destacamos tres resultados de este estudio. Primero, nuestros datos enfatizan que lo que los niños pueden hacer depende del contexto del problema y de las herramientas que los estudiantes tienen a su disposición. En segundo lugar, nuestros datos ilustran cómo algunas representaciones ayudan a los niños con problemas que incluyen una composición de dos transformaciones pero no necesariamente con problemas de transformación entre dos medidas. Además, los niños pueden responder a algunos problemas con mayor éxito cuando pueden interactuar con ellos mediante el uso de representaciones específicas. En tercer lugar, algunas representaciones facilitan una atención explícita a los tipos de cantidades y a la estructura del problema. También discutimos implicancias para la enseñanza de las matemáticas.

Palabras-claves: matemáticas primarias, problemas aditivos, representaciones

Utilisation des tableaux par les enfants du début de l'école élémentaire qui résolvent les problèmes additifs de Vergnaud

Cet article décrit les réponses de jeunes enfants du primaire aux problèmes additifs qu'ils résolvent à l'aide de représentations papier-crayon, en particulier avec l'utilisation d'un tableau. Nous explorons la question de recherche suivante : de quelle manière les tableaux influencent-ils la précision et la capacité des jeunes élèves (de la 1re à la 3e année d'école élémentaire, âgés de 6 à 8 ans) à résoudre des problèmes additifs, notamment en identifiant les inconnues et les composantes des problèmes ? Des enfants d'une école publique située dans une banlieue diversifiée du nord-est des États-Unis ont été interrogés individuellement. Chaque enfant s'est vu présenter six problèmes additifs tirés des travaux de Vergnaud (1982). Nous avons conçu trois contextes de représentation (avec une feuille de papier vierge, avec un tableau sans étiquette ou avec un tableau étiqueté), auxquels les enfants ont été assignés au hasard. Nous obtenons trois résultats. Premièrement, nos données soulignent que, ce que les enfants peuvent faire dépend du contexte du problème et des outils dont ils disposent. Deuxièmement, nos données illustrent comment certaines représentations aident les enfants à résoudre les problèmes de type « composition de deux transformations », mais pas nécessairement les problèmes de « transformation de mesures ». De plus, les enfants sont capables de répondre avec plus de succès à certains problèmes lorsqu'ils sont capables de s'y engager grâce à l'utilisation de représentations spécifiques. Troisièmement, certaines représentations favorisent une attention explicite aux types de quantités et à la structure du problème. Les implications pour l'enseignement sont également discutées.

Mots-clés : mathématiques élémentaires, problèmes additifs, représentations

Background and rationale

In this paper, we focus on young children's work with additive problems when using tables. We argue that children's abilities are dynamic and contextual. Instead of emphasizing what children can and cannot do, we specifically focus on what children can do as dependent on a context that is defined according to the structure and complexity of the problems presented to them and the representations, in this case tables, they use. We provide evidence from an exploratory study with children in Grades 1 through 3 (approximately 6-8 years of age) and explore the following research question: *In what ways do tables influence young students' accuracy and ability to work with additive problems including identifying unknowns and components of the problems?*

To answer this question, we build on Vergnaud and Durand's work (1976) and further investigate the differences between two categories of additive relationships (*state—transformation—state* [STS] and *transformation—transformation —transformation* [TTT]) to compare how children use tables in the two different problem contexts. We focus on tables because we view them as an important, and at the same time understudied, way for children to organize and represent their ideas. We also note that tables are a familiar representation that are used throughout elementary school. Thus, we view them as a tool that is already in children's "toolboxes," and not as a new representation. In our prior work we have found that tables served as an organizational tool that supported students, ages five to nine, in problem solving in the context of algebra (Brizuela & Alvarado, 2010; Brizuela *et al.*, 2021; Brizuela & Lara-Roth, 2002). For example, tables can help students organize co-varying quantities (e.g., (1, 2), (2, 4), and so on for the relationship $y = 2x$).

Children's understandings and construction of additive structures

Piaget's work (1970), as well as other work in mathematics education that has built on his research (Kamii, 1985, 1989; Steffe *et al.*, 1988; Carpenter *et al.*, 1989; Ginsburg, 1977; Vergnaud, 1982), have detailed both the natural abilities of young children as well as the ways in which their initial understandings about number, quantities, mathematical properties, and mathematical relationships, for instance, build on each other through processes of empirical and reflective abstraction (Piaget, 1977). In this paper, given that our starting point was the work of Vergnaud on additive structures (1982), a constructivist framework was fundamental.

Vergnaud's (1982) classification of problems in the conceptual field of additive structures, as well as his description of children's ability to deal with these different kinds of problems, set the stage for our study. Vergnaud described six basic categories of additive relationships among quantities, each with several subcategories. In this paper, we focus on the two categories—STS and TTT—for which Vergnaud provides us with empirical results specific to young children. STS relationships involve a transformation operating on a measurement and resulting in a new measurement, while TTT relationships involve two transformations that are composed into a third transformation (Vergnaud, 1982).

In his 1982 paper, Vergnaud reported on a study he conducted with Durand (1976) in which they examined differences between a transformation between two measures (STS) and a composition of two transformations (TTT). In their 1976 study, Vergnaud and Durand described STS problems as accessible and TTT problems as more challenging. Thus, we used STS problems as a baseline to assess our participants' engagement with additive structures. Vergnaud and Durand found that correct responses in both STS and TTT problems increased, in general, with age.

Much of the work on additive structures (Carpenter *et al.*, 2003; Vergnaud, 1982; Vergnaud & Durand, 1976) illuminates children's strategies without looking at representational supports, and much of the research that does explore the role of supports involves the use of manipulatives, or artifacts that students can handle (see Clements, 2000 for a summary of this work). In his 1982 paper, Vergnaud did not describe providing children with, or requesting children to produce, representations for solving problems. However, he did propose two criteria for the efficiency of "symbolic representations" (p. 53). The first is "symbolic representations should help students to solve problems that they would otherwise fail to solve" and the second is that "symbolic representations should help students in differentiating various structures and classes of problems" (p. 53). Vergnaud also highlighted that "these criteria should be used to evaluate different sorts of symbolic systems, at different stages of the acquisition of additive structures" (p. 53). His proposal was that we should take the time to examine different symbolic systems and see what they can symbolize correctly, their limits, and their advantages and inconveniences. Our research question for this study is related to Vergnaud's proposal as we explore the ways in which tables influence children's thinking about different additive relationships.

Literature Review

Vergnaud (1982) described an intervention study with 11- to 13-year-old children on a variety of additive problems in which they used both equations and arrow diagrams to solve additive problems. At the time of the post-test, only a few students were able to use equations correctly after the intervention, but about 40% of them were able to use arrow diagrams correctly to solve the problems. The results of the experiment led Vergnaud to argue that arrow diagrams may be more appropriate at this age than equations for the solution of problems. Vergnaud (1982) also presented Euler-Venn diagrams, transformation diagrams, algebraic equations, vector diagrams, and distance diagrams.

In our study, we worked with younger children (Grades 1 through 3, or ages 6-8), and we did not include an intervention component; but, as Vergnaud did, we wanted to understand what associations there might be between different uses of tables and children's approaches to different kinds of additive relationships.

We consider tables symbolic representations and define symbolic representations in the same way as Vergnaud. He explained they "stand for objects of different mathematical status: elements, operations, relationships, classes, functions… Of course, there may be different sets of symbols referring to different (or to the same) set of objects" (Vergnaud, 1979, p. 268). We also view tables as tools and rely on Hiebert *et al.*'s (1997) definition of tools. They said tools "help students do things more easily or help students do things they could not do alone […] can enable some thoughts that would hardly be possible without them" (p. 53). Before describing tables specifically, we review prior research on representations and tools for working with additive problems.

Tools for working with additive relationships

Research on additive relationships fits into two paradigms: *operational* and *relational* (Polotskaia & Savard, 2018). The operational paradigm is focused on "arithmetic operations and calculation strategies," while the relational paradigm is focused on "quantitative relationships and modeling" (Polotskaia & Savard, 2018, p. 72). Vergnaud's (1982) descriptions and representations of additive structures are focused on identifying states, transformations, and measurements; his approach closely attends to operations, and thus fits into the *operational paradigm*.

Researchers have advocated for the different approaches or some combination of them for various reasons. In general, one is not better than the other, they are simply different ways of interpreting additive structures. For example, an operational approach can support students to understand operations as processes (Nesher *et al.*, 1982), which might help them to relate operations and inverse operations. Carpenter and his colleagues (1989, 1996) did not refer to their approach to teaching arithmetic problems as operational, and they certainly considered quantities and incorporate modeling, but we describe their work as one example of arithmetic problem categorization that is organized around operations, and thus operational. In their work they mapped addition and subtraction to the concepts of joining and separating, respectively, to support students in identifying the appropriate operation in a word problem. On the other hand, a relational approach can help students attend to the relationships between the quantities in the problem.

Specific representations illustrate different aspects within each paradigm. We begin by focusing on the bar method, a representation that supports a relational view of additive problems (Murata, 2008; Ng & Lee, 2005, 2009; Thirunavukkarasu & Senthilnathan, 2014). Polotskaia and Savard (2018) and several other researchers have studied how this representation can support students in solving additive problems. In this method, students begin by drawing a simple rectangle bar, or in some cases bars, and labeling quantitative parts of the bar(s) to represent the problem (Kaur, 2019).

Kaur (2019) described three types of bar representations: part-whole representations, comparison representations, and change representations. The part-whole, or sometimes referred to as the part-part-whole, representation, "helps students work through word-problems that involve relationships between the whole and its parts" (Kaur, 2019, p. 153). The comparison representation helps students determine the difference between two quantities, and the change representation "shows the relationship between the new value of a quantity and its original value after an increase or a decrease" (Kaur, 2019, p. 156).

Kaur and others' work (Murata, 2008; Ng & Lee, 2005, 2009; Thirunavukkarasu & Senthilnathan, 2014) has shown that the bar method supports students in solving addition problems. The caveat is that the bar method is specific to arithmetic and not otherwise introduced to students, whereas other tools, such as tables, are introduced in multiple mathematical and non-mathematical contexts, and are therefore socially ubiquitous.

Here we summarize Vasconcelos's work (1998), who carried out an investigation that involved the use of some of Vergnaud's additive problems, within the operational paradigm, in didactical situations and included the use of three different representations: Vergnaud's diagrams (1982), Riley *et al.*'s (1983) part-whole diagrams (Figure 1) and manipulatives.

As with Kaur (2019), the students in Vasconcelos' study (1998) participated in an intervention. Vasconcelos had three groups of 8-year-old students, one for each representation. Each group was provided with a pre-test, a teaching intervention and a post-test. The intervention consisted of solving addition and subtraction problems using one of three different representations: diagrams, part-whole diagrams and manipulative materials. Figure 1 shows the representations that Vasconcelos (1998) provided for students when reasoning about this problem: *Carlos collects keychains. Carlos had 8 keychains. His mother gave him 6 keychains. How many keychains does Carlos have now?*

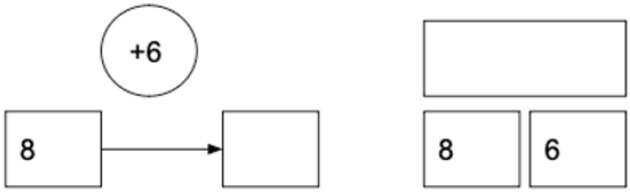

Figure 1. – Vergnaud's (1982, left) and Riley *et al.*'s (1983, right) examples of representations for $8 + 6 = x$ as shown in Vasconcelos (1998, p. 66)

Vasconcelos (1998) found that while all three groups showed increases in the number of correct responses from pre- to post-test, it was those children using Vergnaud's diagrams who were able to achieve the greatest increases in the number of correct responses, and the group that used manipulative materials achieved the smallest increase in the number of correct responses. Vasconcelos concluded that the three different tools that she used in her research were not equally effective. Vasconcelos' findings resonate with Vergnaud's point about how some representations may be more helpful for students than others and our framing of the affordances of tools as linked to the student, tool, and context. In addition, depending on how the students used the representations they may be illustrating aspects of the relational or operational paradigm (Polotskaia & Savard, 2018).

Willis and Fuson (1988) drew similar conclusions about students' use of four representations, which they refer to as schematic drawings, that were selected by the researchers and paired with specific addition and subtraction problem types. Their representations were the same as the ones used by Vasconcelos (1998) and shown in Figure 1. However, they did not use manipulatives, and they added descriptors, such as "start," "change," "part," or "all" to the word problem that was being modeled in the representation. That is, they used the representations shown in Figure 1 but added "S" above the rectangle containing eight to indicate that was the "start" quantity. They added "C" above the circle containing plus six to indicate that was the "change" quantity. To describe the right side of Figure 1, Willis and Fuson (1988) referred to "part" and "all." In the representation shown on the right side of Figure 1, eight and six are both parts and had "P" above them, while the missing quantity had "A" to indicate "all."

Willis and Fuson's (1988) goal was to see if these diagrams supported students in solving the problems. Similar to Kaur (2019) and Vasconcelos (1998), they carried out an intervention because students had not yet been introduced to these kinds of representations. Based on the pre- and post-test results, the intervention proved successful in teaching students how to solve the problems. However, their findings show that sometimes the representations supported students in correctly solving the problems, while other times students were using the diagrams incorrectly or selecting the incorrect diagram for the problem type. One main takeaway was that students struggled to solve the subtraction problems. We hypothesize this finding might indicate more about the representation than about the problem type or student. Perhaps, the representations used to model the subtraction problems were not helpful for some, if not most, students because the representations neither highlighted the operation that should be used to solve the problem nor the relationships between the quantities.

Tables as tools for representing mathematical relationships

We argue that tables can align with both the relational and operational paradigms, depending on how students interpret them and the problem context. In the case of additive problems, a table, labeled or unlabeled, is a space for students to organize quantities. The students might organize the quantities in a way that highlights the relationship between them, or the students might organize the quantities in a way that brings the operation to the forefront. Tables are commonly found in elementary mathematics curriculum, and while other representations, such as the bar model, may be fruitful ways to show mathematical ideas, they are also new to most students.

Before we explore how new representations may support students' learning, we first aim to understand the role of existing, broadly used representations. As a first step, we were curious to know what students might do, and how tables might support or hinder them in solving additive problems. A few studies have explored tables as tools, mainly in the context of early algebra, and the general consensus is that tables support students' mathematics learning. Here we summarize some of that work.

Brizuela *et al.* (2021) found that after participating in a classroom teaching experiment (CTE), Max, a 5-year-old student, could use tables as an organizational tool and to generalize about functional relationships. Through the CTE, Max interacted with tables in various forms including using them to organize co-varying quantities and make observations about patterns and relationships among those quantities. Brizuela and Lara-Roth (2002) also investigated students' use of tables, without the instructional aspect of the Brizuela *et al.* (2021) study and found that without instruction on constructing or using a table, 7-year-old students could use tables to solve problems of an algebraic nature. Additionally, students self-generated tables with no imposed structure.

Schliemann *et al.* (2007) also structured tables to encourage children to attend to the functional relationships between quantities. In their study, the tables presented to the students had incomplete cells. Children had to complete the table following the same function that explained the relationship between other numbers already present in the table. By including in the tables presented to students numbers in the input column that were not in an ordered sequence or that skipped over numbers in a sequence, children were discouraged from simply going down input and output columns to find the patterns of numbers that worked, and instead focused on the function that led the quantity in the input column to transform into the quantity in the output column. Similar to our work, the tables in Schliemann *et al.*'s (2007) study were structured as tools that could allow children to explore relationships among quantities.

Martí (2009) considered the use of tables among students in Grades 2 and 5 in contexts that are more general than algebra, involving organizing sets of data. He found that, "the process of table construction can change the subject's prior knowledge" (p. 145). Martí found that when asked to note a set of categorical data, Grade 2 children never used a table spontaneously, and tables were rarely used by Grade 5 children. Martí argued that primary school children need to be introduced to tasks in which knowledge is expressed in different forms and that require different representations "suited to the demands of the task" (p. 147). He also noted that interpreting tables is cognitively different from constructing and using tables, a distinction that is relevant to our study because with one group of students we scaffolded the construction of tables by providing labels.

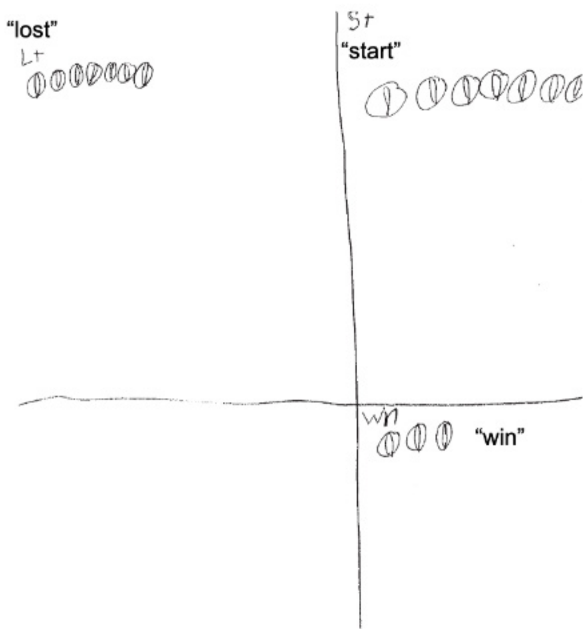

Figure 2. – Gabriela's written work for Bruno ($y - 7 = 3$; TTT) problem
Brizuela & Alvarado, 2010, p. 43.

Brizuela and Alvarado (2010) highlighted as noteworthy Gabriela's unprompted construction of a table to support her in correctly solving the two TTT problems presented to her. Before starting both problems, Gabriela drew a line down the middle of her paper and organized her right column into three labeled sections, each for one quantity in the problem. Figure 2 shows her response to a TTT problem in which students find *a*, knowing *b* and *c*. The problem could be expressed algebraically as $y - 7 = 3$ (Problem 6 from Table 1). Here she used the labels "st" (for Start), "wn" (for win), "eD" (for end), and "Lt" (for Lost). She drew 7 marbles in the "Lt" cell, and then drew 3 marbles in the "eD" cell. After doing this, she added 7 and 3, said the answer was 10, and drew 10 marbles in the "st" cell. While this student may have used "st" to represent a transformation instead of a state, we view this as a productive representation, for this student, in this problem context.

The authors noted that Gabriela's representations were a welcome surprise since, unbeknownst to her, her representations reflected the study design by including a table and labels even though she had only been provided with plain paper and pencil. Her representation provided a structure for the problem and seemed to support her in attending to the kinds of quantities involved in the problem, which aligns with the operational paradigm.

Brizuela and Alvarado (2010) focused exclusively on correct and incorrect responses among the Grade 1 students and reported that for STS problems, which are simpler, Grade 1 children had a higher rate of success when they were able to solve them orally, without using a table. However, for TTT problems, which are more complex, children performed better when they were given the opportunity to produce a table than when they had to solve the problems orally. These initial findings motivated us to further analyze our data and understand the affordances of tables. In this paper, we include data from Grades 2 and 3 as well as Grade 1, in addition to the frequency of correct and incorrect responses.

Rationale for studying the affordances of tables

We focus on tables because very few studies have presented students with tables as a support for solving additive problems, and we were curious to explore the affordances of tables in the context of an additive problem. We were curious about tables, specifically, because they are one of the first canonical mathematical representations with which students are presented. Tables are one of the representations that continuously resurface in students' K-12 education, beyond just mathematics education, so students use and experience them in a variety of school contexts. Tables appear in non-school contexts as well–they are a ubiquitous tool, used in a variety of contexts and in the general media, yet we know very little about how students use them to solve additive problems.

We also focus on tables because we view them as a way to organize additive problems that supports us in understanding what components students are attending to. This approach aligns with the operational paradigm (Polotskaia & Savard, 2018). Seeing what components of a problem students are attending to helps us understand their thinking, but it also allows us to see if students' views of the structure of additive problems parallels that of Vergnaud (1982). That is, do tables support students in organizing transformations and states of additive problems?

We focus on tables instead of other representations, such as arrow diagrams (Vergnaud, 1982) or bar representations (Kaur, 2019), because in order to set up a table, students do not need to determine how quantities are related or to map the quantities to another representation, such as a number line. Our focus was on the initial process of solving an additive problem and to better understand when and how students identify quantities in additive problems. Also, as noted, students are introduced to tables in other mathematical and non-mathematical contexts. Thus, we assume they are more familiar than an arrow diagram or bar representation, for instance.

Additionally, we used 4×2 tables because they provide a structure that aligns well with STS and TTT problems. Specifically, the two columns can be used two different ways. Students could either reason about a transformation, thinking of the two columns as spaces to show "before" and "after," or students could use one column for labels and the other column for quantities corresponding to those labels. In the labeled table context, we provided students with a table that mirrored the latter. We understand there are different ways to represent these problems with or without tables, but selected this approach because it was mathematically sound, organized, and relied on a representation that is commonly used by elementary students.

We specifically investigated the affordances of tables. An "affordance refers to whatever it is about the environment that contributes to the kind of interaction that occurs" (Greeno, 1994, p. 338). We understand an affordance not as a property of a tool but as a property of the interaction between children (in our case) and tools (Gibson, 1977). The affordances we aimed to uncover result from students, with specific prior experiences, interacting with a table in a certain problem context. We make this distinction because it addresses the constructivist underpinnings of this study as this perspective on affordances accounts for children's prior experiences.

Vergnaud's (1982) claim that representations can support students in solving additive problems, coupled with our prior observations of students using tables to organize data, motivated us to further study students' use of tables with a larger group of students using additive problems. We relied on Vergnaud's framework for categorizing problems because it supported us in understanding the different components of these additive problems and how a problem might be decomposed and represented in a table; it also helped us categorize problems according to their complexity.

Method

Data collection

A total of 45 children (22 Grade 1, 12 Grade 2, and 11 Grade 3) from a public school in a diverse suburb in the Northeast of the United States (US) were interviewed individually during the first semester in the school year. Each interview lasted around 20 minutes and all interviews were videotaped. Children could take as long as they needed on each problem. We also collected all written productions. The written productions and transcripts were the primary data source; however, the video recordings were referenced if clarification was needed.

Each step in the problem was read to each child. After each step, children were asked, "Could you show this on your paper/on your table?". So, for instance, in Problem 3 in Table 1, children were told, "Bernardo plays marbles. He loses 7 marbles. Could you show this on paper/on your table? [Waiting for children to show something on paper/on the table.] At the end of the game, he has 3 marbles. Could you show this on paper/on your table? [Waiting for children to show something on paper/on the table.] How many marbles did he have at the beginning of the game? Could you show this on paper/on your table? [Waiting for children to show something on paper/on the table.]".

Each child was presented with six additive problems taken from Vergnaud's (1982) work (see Table 1). Of the six problems three were STS and three were TTT. The three problems for each category were each from a different subcategory. Namely, we presented students with problems of the form: find *a*, knowing *b* and *c*; find *b*, knowing *a* and *c*; and find *c*, knowing *a* and *b* for both STS and TTT. We used both positive, negative, and opposite signed problems. Our goal in using a variety of problems was to see if students' use of tables varied or remained consistent across problem types.

It is typical for students in the US in Grades 1-3 to be introduced to tables as a way to organize data. TTT type problems are typically not introduced in mathematics curriculum in the US. The STS problems served as a sort of baseline, in which students were likely to be successful, and the TTT type as a new environment in which students may or may not be able to solve the problem. We present the problems in Table 1 and a description of the problems in Table 2.

We designed three representational contexts, to which children were randomly assigned.

At each grade level:

- one third of children used plain paper and pencil;
- one third of children used unlabeled tables (i.e., a blank 4×2 grid; see Figures 4 and 7 in the a priori analysis);
- one third of children used labeled tables (i.e., a 4×2 grid with the following labels in the left cell in each one of the four rows: "Start," "First Round," "Second Round," and "End;" see Figures 5 and 8 in the a priori analysis).

Table 1. – Problems presented to children in this study

Problem Number	Description
1	Pedro has 6 marbles. He plays one round of marbles and loses 4 marbles. How many marbles does he have at the end of the game?
2	Pablo plays two rounds in a game of marbles. In the first round he wins 5 marbles. In the second round, he loses 3 marbles. What happened in the game?
3	Bernardo plays marbles. He loses 7 marbles. At the end of the game, he has 3 marbles. How many marbles did he have at the beginning of the game?
4	Claudio has 5 marbles. He plays a round and after he finishes playing he has 9 marbles. What happened during the game?
5	Cristian plays two rounds of marbles. In the first round he wins 5 marbles. Then he plays a second round. At the end of the game he has won 9 marbles. What happened during the second round?
6	Bruno plays two rounds of marbles. He plays the first round, and then after, the second round he loses 7 marbles. After the two rounds, he has won 3 marbles in total. What happened during the first round of the game?

Problems were presented in the same order for each participant.

Each child first solved Problems 1 (STS) and 2 (TTT) from Table 1 orally, with no paper or pencil or any kind of written support. They were then presented with the rest of the problems in a single representational context to which they had been previously, randomly assigned. We presented each student with the first two problems to have baseline data on their ability to solve STS and TTT problems of the same form (i.e., find c, knowing a and b) without any support. Participants then proceeded to solve the remaining problems in their assigned context. This design allowed us to compare within students (e.g., how one student does orally versus with an unlabeled table) and across students (e.g., most students were able to make the unknown explicit when using a labeled table). All children were presented with the problems in Table 1 in the same order. There were a total of 180 written responses provided by students (45 students × 4 problems each using a representational context).

Table 2. – Characteristics of problems presented to children in this study

Problem	Category (Vergnaud, 1982)	Problem equation	Subcategory (Vergnaud, 1982)
1	STS (solve orally, no pencil & paper)	6 – 4 = (2)	find *c*, knowing *a* and *b*
2	TTT (solve orally, no pencil & paper)	5 – 3 = (2)	find *c*, knowing *a* and *b*
3	STS	(10) – 7 = 3	find *a*, knowing *b* and *c*
4	STS	5(+4) = 9	find *b*, knowing *a* and *c*
5	TTT	5(+4) = 9	find *b*, knowing *a* and *c*
6	TTT	(+10) – 7 = 3	find *a*, knowing *b* and *c*

Terms between parentheses in the equations representing the problems are the responses requested from the children.

A priori analysis

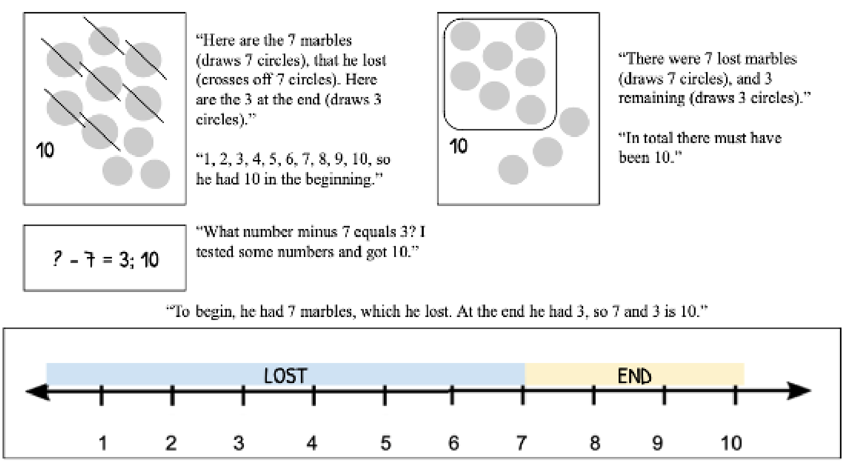

Figure 3. – Four hypothesized strategies for solving Problem 3 (STS, negative transformation, question about initial state) in a paper and pencil context

As part of our *a priori* analysis, we considered ways students might respond to the problems. Our goal here was to make sense of the different problem contexts and how they may or may not support students. Here we present the *a priori* analysis for Problems 3 and 5. We selected these two problems because each student solved them in one of the three representational contexts, versus Problems 1 and 2, which are only solved orally. We selected two problems because we wanted one each of the two types of problems: STS and TTT. Figure 3 shows four hypothesized student responses to Problem 3 (see Tables 1 and 2) in the paper and pencil context. Figure 4 shows hypothesized student responses in the unlabeled table context. The way we intended them to use tables is shown in the third strategy shown in Figure 4, but there are many ways in addition to what is shown that students could use tables to organize their responses.

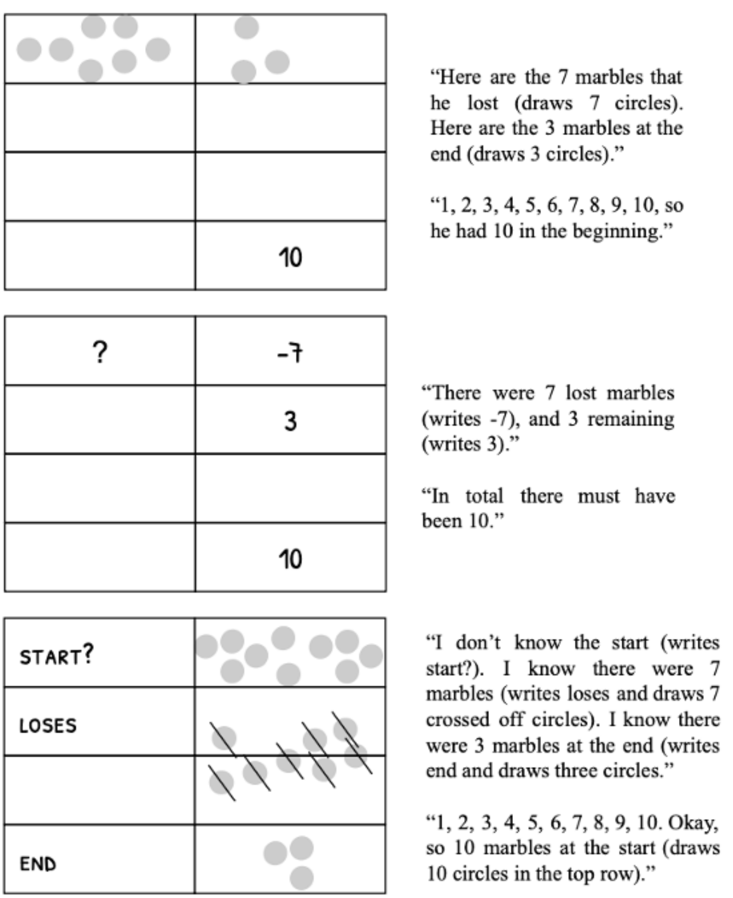

Figure 4. –Three hypothesized strategies for solving Problem 3 (STS, negative transformation, question about initial state) in the unlabeled table context

Lastly, Figure 5 shows how we hypothesized students might use the labeled table to solve Problem 3. Of course, there are other ways students may have used the table, but we did not anticipate as much variation in this context as we did in the paper and pencil and unlabeled table contexts because this context is more structured and provides less flexibility.

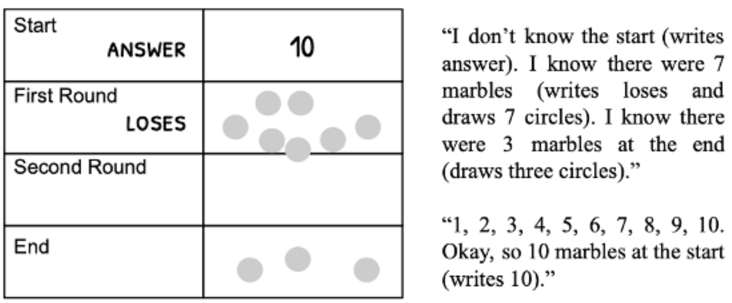

Figure 5. – One hypothesized strategy for solving Problem 3 (STS, negative transformation, question about initial state) in the labeled table context

We conducted the same analysis for Problem 5 (TTT). Figure 6 shows how students might respond to the problem in a paper and pencil context. The example that uses a number line is a combination of a number line focused strategy and a strategy that uses the bar model (Kaur, 2019).

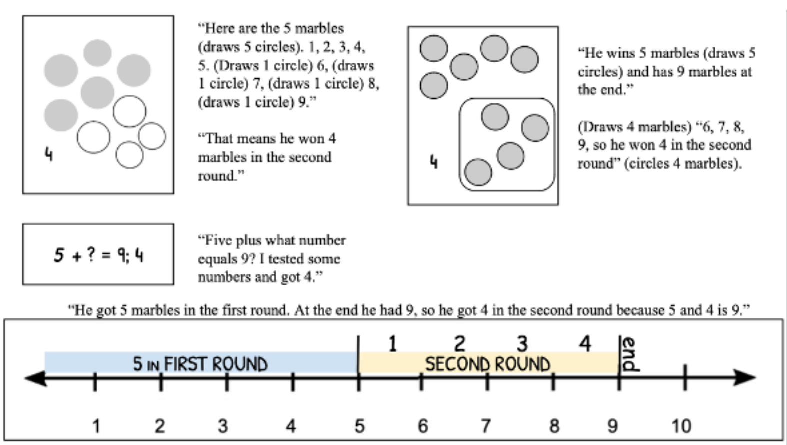

Figure 6. – Four hypothesized strategies for solving Problem 5 (TTT, positive composed transformation, question about one transformation) in a paper and pencil context

Figure 7 shows how students might solve Problem 5 in an unlabeled table context. Here we show one response that is incorrect ("14"), in which the student incorrectly added the marbles won in the first round and the total marbles in the end.

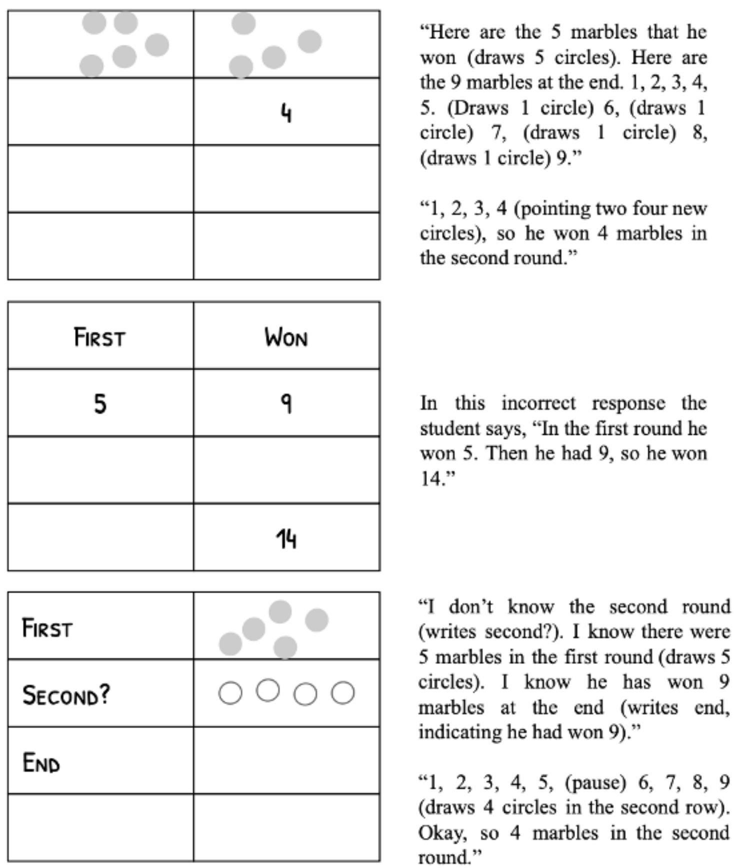

Figure 7. –Three hypothesized strategies for solving Problem 5 (TTT, positive composed transformation, question about one transformation) in an unlabeled table context

Note the accuracy of these responses varies.

Figure 8 shows a hypothetical student's response to Problem 5 in the labeled table context. As with Problem 3, we hypothesized more variety in how students solved these problems in a less structured context, such as paper and pencil. On the other hand, if students used a labeled table, which is more structured, we did not hypothesize as much variety in the way they showed their solution.

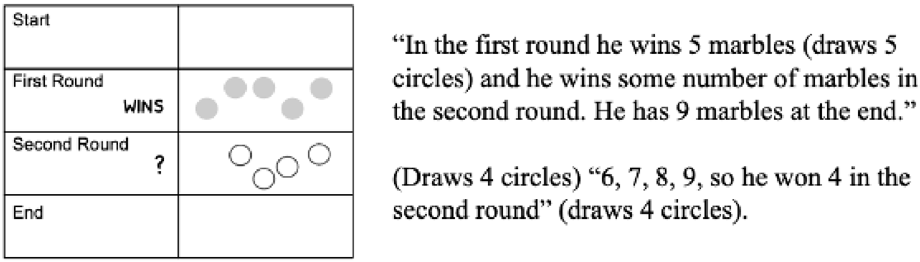

Figure 8. – One hypothesized strategy for solving Problem 5 (TTT, positive composed transformation, question about one transformation) in a labeled table context

Data analysis

We transcribed all interviews. Our analyses were focused on children's written productions and the interview transcripts. We regularly reviewed videos as needed. All problems were analyzed for correctness according to an answer key. Percentages and frequencies of correct and incorrect responses for each problem and representational context were calculated, and all data were recorded in a table, organized by grade, representational context, problem category, and student. Below, in the Results section, we also share examples of two Grade 1 students' responses and two Grade 2 students' responses and describe what their representations might indicate about how they understand the problems.

Results

We begin by reporting on correct and incorrect responses across the different representational contexts as background to then report on the different aspects of the representations produced by children and to explore our research question. Table 3 displays the number of correct responses and the percentage of correct responses.

As expected (Brizuela & Alvarado, 2010) and reported by Vergnaud (1982), more children solved STS problems correctly than TTT problems, and the proportion of correct responses for both problem categories increased with children's age. That is, we found that the percentage of correct responses was higher for STS problems in general, and that, in general, the percentage of correct responses was higher among Grades 2 and 3 children compared to Grade 1 children.

Table 3. – Percentage and number of correct responses

Group		STS			TTT		
		Grades					
		1	2	3	1	2	3
Paper and pencil	oral presentation	67 (6)	100 (4)	100 (3)	11 (1)	25 (1)	33 (1)
	paper and pencil	50 (9)	75 (6)	67 (4)	33 (6)	38 (3)	50 (3)
Unlabeled table	oral presentation	83 (5)	100 (3)	100 (4)	0 (0)	33 (1)	25 (1)
	unlabeled table	67 (8)	83 (5)	100 (8)	25 (3)	50 (3)	63 (5)
Labeled table	oral presentation	71 (5)	100 (5)	100 (4)	29 (2)	0 (0)	0 (0)
	labeled table	29 (4)	100 (10)	75 (6)	0 (0)	60 (6)	75 (6)

As an example, the percentage reported in the first cell of Table 3 (Grade 1, oral presentation) is calculated by multiplying the number of times a Grade 1 student correctly answered a STS problem orally (6) by 100, divided by the total number of responses provided by Grade 1 students in that category of problem (9). The number in parentheses in each cell is the number of participants in that grade multiplied by the number of problems they answered correctly in each of these problem categories and contexts.

In addition to providing further evidence that is consistent with Vergnaud's (1982) findings, Table 3 also compares children's solutions to problems without any kind of written support (i.e., oral presentation) and with written support. We found that, in general, for STS problems, the percentage of correct responses was higher when problems were solved orally. We recognize the placement of the unknown is different in the two STS problems, yet we categorize them together because the underlying structure of the equation is the same. Interestingly, the same was not true for TTT problems. For these more complex problems, the percentage of correct responses was higher when problems were solved with some kind of written support. This is consistent with findings reported in Brizuela and Alvarado (2010).

Furthermore, we notice in Table 3 that in every single case, the proportion of correct responses was higher when children were able to use a table, instead of just paper and pencil. In addition, the proportion of correct responses was higher when children were able to use an unlabeled table in only two cases: for both STS and TTT problems, *only among*

Grade 1 children. In all other cases, for children in Grades 2 and 3, the higher proportion of correct responses was associated with the use of labeled tables. Our interpretation of these differences is that Grade 1 children were not able to fully interact with the labels on the tables due to their still emerging literacy skills. These results highlight a relationship between the type of problem and the specific type of representation that facilitates solving a problem. We elaborate on this point in the discussion.

Representations of unknown quantities

The results of our study also highlight that some representations are more frequently associated with children making unknown quantities explicit in their productions (e.g., by writing "?"). Table 4 shows that in 14 of the 180 written responses produced by children (7.8%), they made the unknown in the problem explicit in some way. Even though these frequencies are very small, it is interesting to focus on some of the patterns that are reflected in the data.

Table 4. – Number of solutions making the unknown in the problem explicit in some way

Group	STS			TTT		
	Grades					
	1	2	3	1	2	3
Paper and pencil	0	0	0	0	1	1
Unlabeled table	0	1	2	2	0	2
Labeled table	0	2	1	0	1	1
Total	6			8		

For STS problems, only Grades 2 and 3 children made the unknown explicit, and only in the unlabeled and labeled tables context. For TTT problems, children in all three grade levels made the unknown explicit, and children in Grades 2 and 3 made the unknown explicit both when using tables as well as when using plain paper and pencil. However, the tables context is the only one in which Grade 1 children made the unknown explicit for TTT problems.

Across both categories of problems, among the older, Grades 2 and 3 children, there were more occurrences of making the unknown explicit in the tables context. This leads us to ask ourselves: *did the structure of the tables facilitate an explicit attention to the unknown in the* *problem*? Additionally, making the unknown explicit was somewhat more frequent in TTT problems than in STS problems, which leads us to wonder: *did problems of more complexity facilitate an explicit attention to the unknown in the problem*? While the frequency of this type of response was very low, when we consider the broader context of what the research literature has described regarding middle school students' work in algebra and the difficulties that have been documented in their use of symbols to represent unknowns and variables (Bednarz, 2001; Bednarz & Janvier, 1996; Booth, 1984; Küchemann, 1981; Vergnaud, 1985; Wagner, 1981), what these young children expressed in their written work is quite remarkable. There is a large body of research demonstrating how young children can use variety of notation to represent unknown quantities; these studies involve classroom teaching experiments and interventions that are carefully designed to support students in using variables (Blanton *et al.*, 2015; Brizuela *et al.*, 2015; Ventura *et al.*, 2021). However, this is not the case in our study, which only involved individual interviews, with no instructional intervention.

Mary, a Grade 1 student, provides us with an example of a response that explicitly attends to the unknown in the problem. In the two STS problems for which she was first given an unlabeled table to work with, Mary distributed the different quantities referred to in the problems in different cells and used another cell to set up the expressions she used to solve both problems. For Problem 3, which could be algebraically expressed as $x - 7 = 3$, she first wrote "– 7" in the top left cell and then 3 and drew three marbles on the top right cell (Figure 9, left). She then wrote $7 + 3 = 10$, and stated that Bernardo started the game with 10 marbles. For Problem 4, which could be expressed algebraically as $5 + x = 9$, Mary wrote the numbers corresponding to the stated amounts in different cells, as well as drawing the appropriate number of marbles next to the numerals (Figure 9, right). Below this, she wrote $4 + 5 = 9$ and stated that Claudio won 4 marbles during the marbles game.

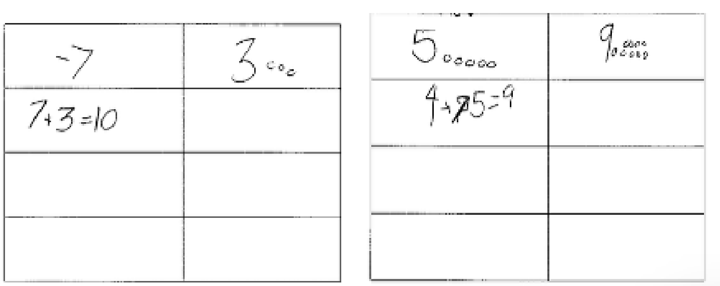

Figure 9. – Mary's written work on problem 3 (STS, negative transformation, question about initial state, $x - 7 = 3$; left) and problem 4 (STS, positive transformation, question about the transformation, $5 + x = 9$; right)

When presented with TTT problems 5 and 6, Mary shifted to making explicit the unknown in the problems, as shown in Figure 10. In problem 5, the problem stated the first transformation of marbles (+ 5) as well as the global transformation (+ 9). What happened in the second round is unknown. Mary represented this by writing a question mark (see Figure 10, left) next to "2nd" (for the second round). Once the whole problem was read, Mary wrote in the answer to the problem, 4 marbles, next to the question mark.

In problem 6, her response was similar. This time, the unknown amount was what had happened in the first round of the marbles game. Once again, Mary wrote a question mark (see Figure 10, right) next to "1st" (for first round). After the rest of the problem had been read out loud to her, she came back to this first cell, wrote in the answer (10), and connected the numeral with the question mark using an arrow (as if saying: 10 is the unknown amount, these two quantities—the unknown quantity and the quantity ten—are connected).

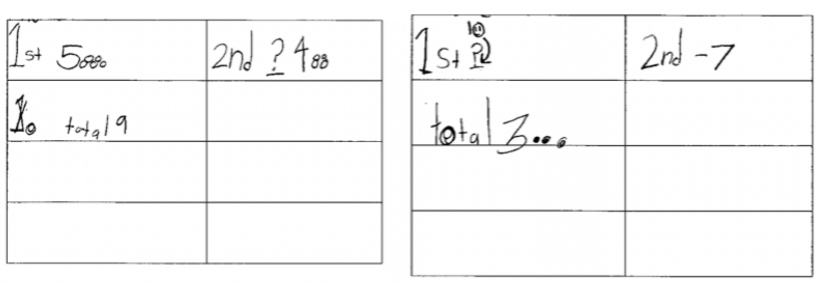

Figure 10. – Mary's written work on problem 5 (TTT, question about one transformation $5 + y = 9$; left) and problem 6 (TTT, question about one transformation $y - 7 = 3$; right)

Mary's case illustrates, in reference to the above questions, that tables may facilitate an explicit attention to the unknown in the problem. Additionally, her case also illustrates that TTT problems could facilitate an explicit attention to the unknown in the problem. However, it may also be that her increasing familiarity with the type of problem presented to her prompted her to begin to focus on the types of quantities involved in the problem.

Making states and transformations explicit

Table 5 shows that in 59 of the 180 written responses (33%) produced by children, they made the state and/or transformation explicit with a word, letter, or symbol (e.g., by writing "has," "got," or "lost").

Table 5. – Frequency of responses that made the state and/or transformation explicit with a word, letter, or symbol

Group	**STS**			**TTT**		
	Grades					
	1	**2**	**3**	**1**	**2**	**3**
Paper and pencil	1	1	4	3	5	3
Unlabeled table	6	2	1	4	3	1
Labeled table	3	5	3	5	7	2
Total	26			33		

For *both* STS and TTT problems, explicit representation of the state or transformation was more frequent in the tables representational context. That is, when provided with tables, children not only made explicit unknown quantities present in the problems, but they also made the state and/or transformations involved in the problem explicit in some way. For STS problems, this was true across both types of tables (with or without labels); for TTT problems, this was true more often for labeled tables.

Grades 1 and 2 students were more likely to make the state and/or transformation explicit for TTT problems, and when using both types of tables. Among Grade 3 students, the picture is a little different. These children tended to make the state and/or transformation explicit evenly across both types of problems, as well as when using both plain paper and pencil and when using both kinds of tables.

The results presented in Table 5 raise again the following question: did the structure of the tables facilitate an explicit attention to the type of quantity in the problem (i.e., state and/or transformation)? In addition, especially for Grades 1 and 2 children, our results also make us wonder if problems of more complexity facilitated an explicit attention to the type of quantity in the problem (i.e., state and/or transformation).

We highlight a few examples from the children's written productions to illustrate the ways in which they made the state and/or transformation in a problem explicit. First, we show the work of two Grade 1 students on problem 3 (expressed algebraically as $x - 7 = 3$). Clara solved the problem correctly (Figure 11), and in doing so, she carefully made sure to notate that 7 was the quantity of marbles that had been "*lost*" in the first round of the game, and in the end 3 marbles is how many marbles Bernardo "*has.*" Her representation makes a lot of sense: had she not notated what the 7 and 3 referred to, she may have ended up carrying out an arithmetical operation that may have been inappropriate for the problem at hand.

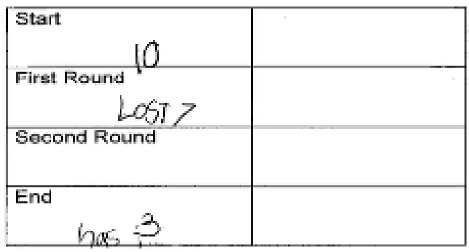

Figure 11. – Clara's work on problem 3 (STS, negative transformation, question about initial state, $x - 7 = 3$)

Callie, also in Grade 1, represented the same problem in a similar way (Figure 12). Since she worked in the unlabeled table condition, she did not have the structure to detail the different steps in the problem. Still, Callie detailed the types of quantities, or transformations that occur in the problem, in the same way that Clara had (Figure 11), by indicating that in the marbles game, 7 marbles have been "*lost*" and that in the end (this remains implicit in her representations) Bernardo "*has*" 3 marbles. Callie did not produce a written representation for the solution to the problem, providing the answer "he started with 10 marbles" orally.

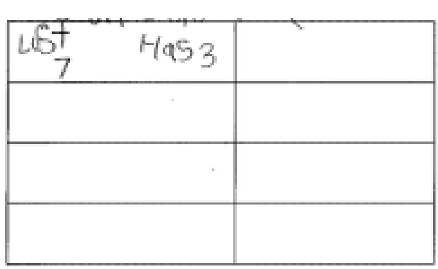

Figure 12. – Callie's work on problem 3 (STS, negative transformation, initial state, $x - 7 = 3$)

Next, we focus on two Grade 2 children working on problem 4, also a STS problem. This problem can be expressed algebraically as $5 + x = 9$. Ezra used letters to indicate the state at each step in the problem, with the support of the structure of the labeled table he was given. He indicated with an "*h*" the amount of marbles that Claudio "*has*" at the start of the game. He did not highlight a transformation or state for the amount of marbles that Claudio had at the end of the game, 9. We assume, therefore, that Ezra was using the label of the table he was given to be able to highlight and qualify what 9 was referring to. He gave the solution to the problem as 4, adding a "*g*" for "*got*." Therefore, unlike Clara, in Figure 11, who just made a representation with the solution to the problem, placing the number 10 in the appropriate cell, Ezra highlighted that the solution to problem 4 was not only 4, but that 4 was the quantity of marbles that Claudio "*got*" during the marbles game (see Figure 13).

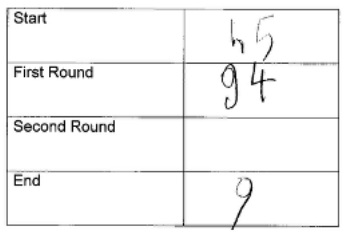

Figure 13. – Ezra's work on problem 4 (STS, looking for the transformation $5 + x = 9$)

First line, Ezra wrote h 5, second line: g 4, fourth line: 9.

Our last example is that of Connor, also a Grade 2 student, working on problem 4 (Figure 14). In his work, Connor made a representation in the first cell indicating that at the start of the marble game Claudio had 5 marbles. As the problem was read to him, he made a representation in the last cell, indicating that at the end of the marbles game Claudio "*has*" 9 marbles. Connor's solution to the problem was that during the game Claudio got 4 marbles. However, he was subtle in the way in which he expressed this in his written response. He wrote "got 2" for both "First Round" and "Second Round"—which together account for the total of "got 4." Our interpretation of his response is that he found a way for his response (got 4 marbles) to match the structure of the table, in which he had the opportunity to state what happened in two different possible rounds of the game. While the final quantitative solution is the same, we speculate that the structure of the table created a scaffold for Connor to think about the problem, providing further evidence for our argument that some representations facilitate an explicit attention to types of quantities and the structure of the problem. We recognize that what is a scaffold for Connor might also be a challenge for other students. That is, they might not know how to use the "second round" cell of the table.

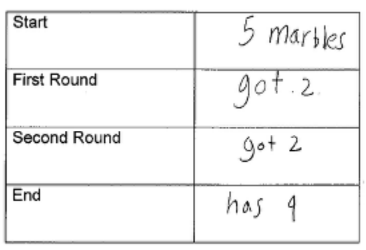

Figure 14. – Connor's work on problem 4 (STS, transformation of states looking for the transformation $5 + x = 9$)

Discussion and conclusions

We set out this exploratory study to respond to the research question: in what ways do tables influence young students' (Grades 1-3; ages 6-8) accuracy and ability to work with additive problems including identifying unknowns and components of the problems? We highlight three findings from this study that relate to this research question.

Problem context and representational tools shape students' thinking

First, our data emphasize that what children can do depends on the context of the problem and the tools available to them. The study we just presented, and the children we showcased, illustrate just how futile it would be to try to characterize their understandings as unitary and uniform entities. Instead, their responses are highly contextual—specifically, as shown in this paper, they depend on the kinds of representations they use when solving problems. In this way, our data support the framing of tables as not having attributes that are affordances of their own, rather the affordances lie in the interaction between the student, the table, and the problem context (Greeno, 1994).

Such a conclusion may suggest that our findings are ungeneralizable and thus, a shortcoming of our study, but we view this as a key takeaway. That is, we highlight that our findings have illuminated some of the nuanced and specific ways that students use tables to support their thinking about additive structures. For example, in our study we observed differences between the ways the Grade 1 students used tables as compared to the Grades 2 and 3 students. Perhaps the added experience of using and writing words to label objects made the latter students more apt to use labeled tables productively.

We also observed students using the table to influence their interpretation of the problem context. For example, Connor, whose work is shown in Figure 14, organized the components of the problem into each cell of the table. The structure of the table (with four rows) led the student to a particular solution that resulted in filling in all four cells. This adaptation highlights that students' solutions are a result of their interpretation of the problem context and the tools available to them. We see this in Connor's use of the representation and assume that because the table had four rows, he concluded that there must be four quantities (two states and two unknowns). Connor's way of representing the problem highlights that students' solutions are a result of their interpretation of the problem context and the tools available to them.

Our observations of Connor's experience highlight the value in understanding how different representations highlight different aspects of the problem context. For example, a solution or representation that would call attention to the relationships between the quantities, such as the bar model (Kaur, 2019), may have better suited the problem context and supported Connor's thinking.

Tables support *some* additive problem solving

Second, our data illustrate how tables might help children with some additive problems, but not necessarily with every additive problem. While being able to use some kind of written support led to higher rates of correct responses than just an oral context with no representational supports, in our study the increased structure of the tables (both labeled and unlabeled) was associated with even higher rates of correct responses. Despite a small sample size, we argue that this is noteworthy because furthermore, the more complex the problem was (in our case, composition of transformation problems), the more likely it was that the greater structure of the tables led to correct responses among the children. Our results suggest that had the students in Vergnaud's study (1982) been provided with written supports, the results he obtained may have been quite different.

While we found in our study that the more structured tools for representing were most helpful when solving the more complex problems, these same tools (labelled tables) were not necessarily as helpful when it came to more simple problems such as transformation between two states. For less complex problems, written supports seemed to get in the way of successfully solving a problem. For more complex problems, the more structured written support provided by a labeled table was more helpful than an oral approach to the problem as well as more helpful than the less structured plain paper and pencil context. We observe that in a small sample, so we cannot claim these trends will always hold true. But it do provide a partial answer to our research question: written supports in general, and tables specifically, are associated with a higher frequency of correct responses in this study. Furthermore, this is especially true for more complex problems such as composition of transformations.

These findings build on Vergnaud (1982) and Vergnaud and Durand (1976) by further confirming the differences of complexity between STS and TTT problems, but also add to the literature on children's additive structures by providing a more nuanced view of the delicate interaction between the complexity of these problems and the specific types of representations that might be most helpful to solve these different problems.

Certain representations highlight certain aspects of additive relationships

The third finding we emphasize is that some tools for representing facilitate an explicit attention to types of quantities and the structure of the problem. This was already illustrated by Brizuela and Alvarado (2010), with their case study of a Grade 1 student, who made her own tables and made explicit the types of quantities that were involved and the different steps in the problem. Similarly, we have shown that an affordance of tables, as opposed to plain paper and pencil, is that children can use them to make unknowns, states, transformations and types of quantities explicit. However, more structured tools for representing, such as tables, also seem to be more helpful for more complex problems, while paper and pencil is helpful enough for more simple problems.

Consider Mary, the Grade 1 student who applied the same strategy to solving the STS and TTT problems with no paper or pencil or written support, but used different strategies for the two types of problems in the table context. Specifically, when presented with composition of transformations problems, Mary shifted to making explicit the unknown in the way she represented the problems in a table.

We see here how students can use tables to distinguish components in a way that aligns with the operational paradigm (Polotskaia & Savard, 2018) versus the relational paradigm, which would involve highlighting the relationships between quantities, not the quantities themselves (see Kaur, 2019 for one example). This finding highlights how certain representations allow children opportunities to focus on nuances of the problems—such as the presence of unknown quantities, and the specific kinds of transformations that are brought up—more than others.

In terms of our theoretical framework, the evidence we have presented lends support to Vergnaud's (1982) criteria for symbolic representations: in the case of composition of transformations problems, correct responses were more likely when children were able to use some kind of representational tool (e.g., paper and pencil or a table). It supports the idea that representing can play an important role in solving problems, amplifying thinking by facilitating children's solving of problems that might otherwise be out of their reach. The fact that making explicit the unknown, the type of quantity, and the step in the problem was more likely to occur in the tables context provides some initial insights into the idea that being able to structure information and data in specific formats, such as tables, can have a positive impact on the understanding of a problem. However, our exploratory study also adds nuance to this framework, by illustrating that tables are not automatically amplifying or helpful—their role is tightly connected to the specificities of the problem, of the task, and of the representation itself.

The implications of this exploratory study are relevant to mathematics education at the early childhood and elementary school levels. First of all, our results suggest that there is a great need to integrate representational tools into all educational activities, as called for by Martí (2009). We focus on tables because for most students they are already part of the elementary mathematics curriculum and ubiquitous in everyday life. Other representations, such as the bar model, are new to most students. Thus, we first aim to understand the role of this existing, broadly used representation.

In addition, there is a great need to pay attention to the ways in which tables might be more or less helpful for specific kinds of problems. We cannot make blanket statements, generalizations, or recommendations about tables. However, we can make the assumption that using tables as tools to represent, seems to help children solve problems in a more productive way. For instance, even the seemingly simple unlabeled table provides some structure for the child, allowing them to assume that information should be organized in some way. The representational space presented to them through the unlabeled table helps them to focus and perhaps ask themselves, "what should or could I put in each one of these empty spaces [i.e., the cells]?" Our data also suggest that an early exposure and familiarity with tables might facilitate children's work on multi-step problems and might help them to attend to the underlying structure of problems.

Our exploratory study should be followed up with others that involve a larger sample and perhaps other types of representations, such as bar diagrams or number lines, as well as different levels of problem complexity, beyond Vergnaud's (1982) types of problems. However, in spite of these limitations, we were able to observe interesting differences across types of representations and types of problems that are important to attend to and follow up with further research.

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