Adding diversity to mathematical connections to counter Klein's second discontinuity

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For instructors that try to make university mathematics courses relevant to future secondary school teachers, doing so generally involves making connections between university mathematics content and school mathematics content–in attempts to counter what Felix Klein referred to as a "double discontinuity." In this paper, I consider the nature of the mathematical connections that bridge these two domains, and common distinctions made in extant literature between them, such as directionality. Through this analysis, I point out another aspect of these connections that has been left implicit: university mathematics is primarily–and reasonably–framed as a superset of school mathematics content. In this paper I consider alternatives, in particular conceptualizing connections that invert this typical relational connection–i.e., a subset relational connection–and I exemplify these connections with concepts from university courses such as real analysis and abstract algebra. Then, I consider the rationale for doing so in terms of secondary teacher education, and the ways that diversifying our framework of connections in this way can be used to help counter Klein's second discontinuity.

Keywords: university mathematics, mathematics teacher education, mathematical connections

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Ajouter de la diversité aux connexions mathématiques pour contrer la seconde discontinuité de Klein

Pour les enseignants qui tentent de rendre les cours de mathématiques universitaires pertinents pour les futurs enseignants du secondaire, cela implique généralement d'établir des liens entre le contenu des mathématiques universitaires et celui des mathématiques scolaires, dans une tentative de contrer ce que Felix Klein appelle une « double discontinuité ». Dans cet article, j'examine la nature des connexions mathématiques qui relient ces deux domaines et les distinctions courantes faites entre eux dans la littérature existante, telle que la « directionality ». À travers cette analyse, je pointe un autre aspect de ces connexions mathématiques qui est resté implicite : les mathématiques universitaires sont principalement – et raisonnablement – présentées comme un sur-ensemble du contenu des mathématiques scolaires. Dans cet article, j'envisage des alternatives, en particulier la conceptualisation de connexions mathématiques qui inversent cette connexion relationnelle typique – c'est-à-dire une connexion relationnelle de sous-ensemble – et j'illustre ces connexions avec des concepts issus de cours universitaires tels que l'analyse réelle et l'algèbre abstraite. J'examine ensuite la justification de cette démarche en termes de formation des enseignants du secondaire et les façons dont la diversification de notre cadre de connexions mathématiques peut ainsi être utilisée pour aider à contrer la seconde discontinuité de Klein.

Mots-clés : mathématiques universitaires, formation des enseignants en mathématiques, connexions mathématiques

Agregar diversidad a las conexiones matemáticas para contrarrestar la segunda discontinuidad de Klein

Para los profesores que intentan que los cursos de matemáticas universitarias sean relevantes para los futuros profesores de secundaria, hacerlo generalmente implica establecer conexiones entre el contenido de matemáticas universitarias y el contenido de matemáticas escolares–en un intento de contrarrestar lo que Felix Klein denominó una "doble discontinuidad." En este artículo, considero la naturaleza de las conexiones matemáticas que unen estos dos dominios y las distinciones comunes que se hacen en la literatura existente entre ellos, como la direccionalidad. A través de este análisis, señalo otro aspecto de estas conexiones que se ha dejado implícito: las matemáticas universitarias se enmarcan principalmente (y razonablemente) como un superconjunto del contenido de matemáticas escolares. En este artículo considero alternativas, en particular la conceptualización de conexiones que invierten esta conexión relacional típica–es decir, una conexión relacional de subconjunto–y ejemplifico estas conexiones con conceptos de cursos universitarios como análisis real y álgebra abstracta. Luego, considero la lógica para hacerlo en términos de la formación de profesores de secundaria y las formas en que diversificar nuestro marco de conexiones de esta manera puede usarse para ayudar a contrarrestar la segunda discontinuidad de Klein.

Palabras-claves: matemáticas universitarias, formación de profesores de matemáticas, conexiones matemáticas

The international Teacher Education and Development Study in Mathematics (TEDS-M) highlighted the great diversity in mathematical requirements in teacher education programs, but also found most secondary teacher education programs usually require students to take at least some university mathematics courses (Ingvarson et al., 2013). Although this requirement is not universal, taking such mathematical coursework, offered by university mathematics departments, is reasonably common in secondary teacher education. And yet such courses are not without significant challenges (e.g., Wasserman et al., 2018; Zazkis & Leikin, 2010). One challenge is related to Felix Klein's (1932/2016) description of the "double discontinuity" teachers face in their mathematical preparation–which captures the gap future teachers face as they transition *to* their university studies (Klein's first discontinuity), and then again *from* their university studies (Klein's second discontinuity). Broadly speaking, Klein's resolution involved elaborating mathematical connections between school and university mathematics, which were captured in a series of volumes entitled "Elementary mathematics from a higher standpoint" (Klein, 1932/2016; Weigand et al., 2019). Various scholars–indeed, all those contributing to this seminar series and special journal issue–have continued developing this line of work, as well as expanding it, to consider how to help make university mathematics courses more relevant to teacher preparation (e.g., Alvarez et al., 2020; Cho & Kwon, 2017; Derouet et al., 2018; Dreher et al., 2018; Gueudet et al., 2016; Stylianides & Stylianides, 2014; Wasserman, 2018; Wasserman & McGuffey, 2021; Planchon, 2019; Winsløw & Grønbæk, 2014). Particularly with respect to Klein's second discontinuity, one important theme in the literature has been identifying the kinds of connections to make explicit, as well as differentiating various types of connections. In this theoretical paper, I summarize recent literature and then exemplify a new distinction to consider, which provides opportunities to generate other mathematical connections; then, I make an argument about why these new types of connections might be especially useful with respect to secondary teacher education, as well as how they might be incorporated into university mathematics courses.

Literature

In this section, I begin with Klein's work and elaborate on developments in the field that have expanded the kinds of connections one might consider with respect to Klein's second discontinuity. Notably, this work has involved differentiating mathematical and didactical (which I use roughly to mean mathematics-specific pedagogical) connections, as well as top-down and bottom-up connections. I use examples of these different types of connections from the literature to situate and exemplify another dimension to consider in distinguishing mathematical connections.

Klein's Second Discontinuity

Before discussing the second discontinuity, it is important to briefly describe the first. The first discontinuity Klein described was about the transition from school mathematics (SM) to university mathematics (UM). Primarily, this describes the abrupt transition students sometimes feel when they encounter more abstract, proof-based mathematics courses at the university, which are quite different than their school studies. As one example, university mathematics courses such as abstract algebra often do not resemble, nor develop from, the school algebra students know. There are other sociocultural transitions between these differing institutional contexts, but the epistemological and cognitive transitions in terms of the mathematics itself often feel like discrete jumps (Di Martino, Gregorio, & Iannone, 2023; Gueudet et al., 2016). Work in this area primarily aims to identify ways to smooth over these gaps–meaning, to coordinate better alignment and a clearer progression throughout the transition, perhaps with "bridging courses," and to connect the university content to the school mathematics students already know.

Klein's second discontinuity is different–more specific to future secondary teachers and, fundamentally, about teacher preparation. For those studying university mathematics who plan to become school teachers, a second discontinuity is experienced as they return to the school mathematics they will teach and grapple with how their university studies relate to the tasks of teaching. Empirical research suggests this is particularly challenging (e.g., Goulding et al., 2003; Hoth et al., 2019; Ticknor, 2012; Wasserman et al., 2018; Zazkis & Leikin, 2010). Although there are a variety of dimensions to this transition–e.g., the institutional context, the subject's differing role as teacher versus student, the temporal gap, etc. (Wasserman et al., 2017; Winsløw & Grønbæk, 2014)–at some level this second discontinuity asks two different kinds of questions: i) How might university mathematics change how secondary teachers understand the school mathematics they will teach; and ii) How might university mathematics inform didactical ideas for how to teach this school mathematics?

Regarding the first question, one might consider this in relation to the notion of backward transfer. Hohensee (2014) defined backward transfer as "the influence that constructing and subsequently generating new knowledge has on one's ways of reasoning about related mathematical concepts that one has encountered previously" (p. 136). He found that it was possible to productively influence learners' ways of reasoning and conceptual connections to prior content while learning new material. Such backward transfer–from UM to SM–is a desired mechanism for helping counter Klein's second discontinuity. It provides a rationale for how study of UM might contribute to teacher's mathematical formation. In terms of this mathematical knowledge, Skemp's (1979) differentiation–which is related to Piaget's (1952) notions of assimilation and accommodation–between 'expansion of cognitive structure' (e.g., adding cases to existing structure) and 'mental reconstruction' (e.g., a reorganization of structures) provides additional insight into how backward transfer might influence one's conceptions and ways of reasoning. In relation to the teacher education literature, Wasserman's (2018) knowledge of nonlocal mathematics for teaching described this as a *mathematically powerful understanding*–not just any mathematical connection from UM, but particular connections that change and reshape one's perception of, or understanding about, the SM they will teach.

The second question stems from the fact that efforts focused solely on mathematics have not had the fully desired effect. That is to say, the assumption that studying UM will somehow "trickle-down" to inform teaching practice, in the terms of Wu (2011), has not been supported by empirical evidence (e.g., Hoth et al., 2020; Zazkis & Leikin, 2010). Recent emphases on practice-based approaches to teacher knowledge in the teacher education literature (e.g., Ball, Thames, & Phelps, 2008; Shulman, 1986) seem to support the difficulties that are evident in empirical studies, as do notions of 'far transfer' (e.g., Wasserman et al., 2019). In other words, it is a fundamentally different question to ask how knowing the set of invertible functions under function composition is a group influences one's mathematical understanding of inverse functions, than it is to ask how such knowledge influences one's approach to teaching secondary students about inverse functions. Meaning, backward transfer in mathematical knowledge may be necessary but insufficient; a connection to teaching practice is another potent mechanism to help counter Klein's second discontinuity. Wasserman's (2018) knowledge of nonlocal mathematics for teaching described this as a *pedagogically powerful understanding*–that the mathematically powerful understandings from UM actually shape and influence classroom teaching practice in some way.

In the introductory survey paper for a recent special issue in *ZDM–Mathematics Education*, Wasserman, Buchbinder, and Buchholtz (2023) surveyed the literature to identify and depict theoretical distinctions with regard to university mathematics courses as they relate to secondary teacher preparation. They captured three distinctions with a figure that identified (i) two planes–the first *mathematical* and the second *didactical*; (ii) within each plane was a collection of mathematical or didactical concepts–differentiated between *school* and *university*; (iii) underlying each plane of *concepts* were foundational mathematical or didactical *practices and beliefs*. Briefly, to connect these to the literature base, the first distinction is one between a scientific discipline and the didactics of its school subject–concerned with the preparation of content for students (e.g., Winkelmann, 1994); within the Anthropological Theory of the Didactic (ATD) (Chevallard & Bosch, 2020) this might be understood as a distinction between mathematical and didactical praxeologies. The second one connects to what Gueudet et al. (2016) referred to as the "school level" when investigating mathematical transitions–which in ATD would relate to the institutional context. The third relates to the distinction between content standards about particular mathematical concepts (e.g., the quadratic formula) and process standards about mathematical processes and activities (e.g., representing, reasoning and proof) (cf., NCTM, 2000); in ATD, it can be related to the distinction between a logos block and praxis block. Within the current conversation, these different theoretical distinctions help frame the challenges of (and opportunities for) countering Klein's second discontinuity. There is a disconnect between school and university mathematics on the mathematical plane, which requires attention, as well as a gap between mathematics and teaching mathematics (evident from the two distinct mathematical and didactical planes) that needs to be bridged.

Throughout the literature, the most prominent idea present for addressing Klein's double discontinuity has to do with making *connections*–essentially, a relation that links one thing to something else. For the purposes of our discussion, the "things" being linked in this context refer essentially to triples composed from the three dichotomous distinctions described above (e.g., school mathematical concepts). Given the importance of school and university mathematics to these connections, I first elaborate further on this distinction and its conceptualization in this paper; then, I move onto connections between them and others, describing some of the various types and dimensions of connections in extant literature as they relate to countering Klein's second discontinuity.

Conceptualization of School Mathematics

Although university mathematics is often a synonym for academic mathematics, broadly speaking, school mathematics (SM) tends to be constituted by the mathematics discussed in primary and secondary education (K-12), prior to university studies, as specified by various curriculum documents–such as national standards, textbooks, assessments, and so forth. Notably, curriculum documents differ by countries and local contexts, which means SM is not universal. Yet, even without such contextual differences, there can still be room for disagreement. Namely, whether SM is constituted by the concepts as specified, or by the examples that get used. Either might be reasonable.

Consider the following: suppose "polynomial functions" is on the list of concepts to be studied; and suppose a student has been given the definition and has seen examples up to some basic cubics. Does SM in this case include, for example, quintics? On one hand, it's reasonable to say yes, given there has been a general definition that would include quintics, and the fact that one cannot reasonably exhaust all examples–some will always be missing (if not quintics, then perhaps septics). On the other hand, it's reasonable to say no, given that students who have only seen cubics might be genuinely confused about quintics and their functional behavior, having never seen one. Now having made the case both routes could be sensible, I now give what I will mean by SM in this paper and a justification for this choice.

For this paper, I take SM to stand for the collection of concepts studied, as given by the definitions in the various curriculum documents such as school standards or textbooks (and not just the collection of examples used). This of course varies by location. Let's consider the concept of function–a relevant example for this paper–and begin by looking at the Common Core State Standards from the United States (CCSSM, 2010). One standard specifies: "Understand that a function is a rule that assigns to each input exactly one output" (8.F.1); in another place, it is stated: "Functions describe situations where one quantity determines another" (p. 67), and "In school mathematics, functions usually have numerical inputs and outputs…" (p. 67). One might question whether the function concept described is equivalent to "real-valued functions" (e.g., between quantities); if so, the conceptual scope would be limited accordingly. But the definition and the word "usually" seem to suggest a more abstract notion–at least in the CCSSM. A grade 10 Canadian textbook, published by Pearson, defines function in terms of relations: "A function is a special type of relation where each element of the domain is associated with exactly one element in the range" (Davis et al., 2010, p. 265), and the examples in that book make clear the domain and range need not be numerical values. That is, the concept is defined to include functions on non-numerical, abstract sets of objects. In my meaning of SM, then–regardless of whether a student in these two contexts is ever introduced to examples of non-numerical functions–the SM concept of function (in these two specific cases) includes functions on abstracts sets of objects. (Notably, other countries or contexts might in fact limit the SM concept of function to real-valued functions, continuous functions, or something else.)

By making this choice about how to conceptualize SM, I want to clarify two things. First, simply because a defined SM concept allows room for particular examples does not mean I am advocating such examples be given to SM students. As an illustration, the existence of a piecewise function–or, as a parallel to the prior example, a 24th degree polynomial–does not mean I would advocate an explicit example should be, or would have to be, given in order for it to be part of the relevant SM concept. Second, this rationale is premised, to some degree, on the idea that teachers play an important role in the curriculum process; that is, the choice to use–or not to use–particular examples with particular classes of students comes down to a professional judgement. And such judgement is based on informed decision-making, i.e., recognizing that, at least in the two contexts presented, functions between non-numerical sets are in fact possibilities.

Mathematical Connections

I begin with *mathematical connections*. In this context, a mathematical connection is *a relation between mathematical concepts or practices encountered in university mathematics (UM) and those encountered in school mathematics (SM)*. In short, it is an arrow representing a link on the mathematical plane between UM and SM. For instance, pointing out that the addition of integers in school mathematics is an example of a group $(\mathbb{Z}, +)$ studied at university would be one such connection.

Klein's own approach to counter the double discontinuity, "elementary mathematics from a higher standpoint," was essentially about elaborating mathematical connections between concepts. His goal was to point to the fundamental coherence (not disconnectedness) of mathematics, and to demonstrate how school mathematics could be understood in relation to university mathematics. For example, Klein (1932/2016) pointed out that drawing two-variable functions $f(x,y) = 0$ as level curves in the xy-plane, as studied in university, can be connected to solving one-parameter equations like $f(x) = k$ as done in school mathematics by finding intersections of the curve with the line $y = k$. In addition to being essentially mathematical, the bulk of Klein's connections also had another commonality; they primarily started from university mathematical concepts, which were presumed to be known by students, and pointed to how the school mathematics could be viewed and further understood through this lens. Although this does not appear to be an explicit aim of Klein's, in addition to mathematical concepts, a mathematical connection might also relate mathematical practices in university and school mathematics.

It is important to note that, although there are points of connection, there also tend to be fundamental distinctions of both concepts and practices between UM and SM. For example, UM tends to focus more on proving as a justification process (rather than supplying reasonable evidence); Dreyfus (1991), for instance, characterized university mathematics in terms of emphases on activities such as generalizing, synthesizing, abstracting, defining, and proving. The key point is that, although clearly similar, mathematical practice as it happens in UM can have a slightly different feel than the practices of SM. Similarly, the concepts studied tend to be somewhat different–namely, in UM, they are more abstract (e.g., Gueudet et al., 2016; Tall, 1991). A group, for example, is a more abstract structure than elementary arithmetic with integers. This abstractness of concepts is similarly reinforced by Dreyfus' emphasis on abstraction as a university mathematical practice. University topics like multi-variable functions, or groups, tend to be further abstractions and generalizations of similar school concepts like single-variable functions, or basic arithmetic; meaning, there tends to be a particular relational structure–which is a point to which I will return.

Mathematical Connections: Top-down and Bottom-up directions

In addition to Klein's work, various scholars have built on his work by trying to complement it with "higher mathematics from an elementary standpoint" (e.g., Courant & Robbins, 1962; Spiegel, 1950; Mosquera, 1992). In some sense, this is about a starting point–what is presumed to be known. In this complementary context, one uses relatively elementary techniques and concepts to gain access to some more advanced mathematical concepts–i.e., we view UM through the lens of SM. This complement to Klein has been captured in the literature as a contrast between a top-down (Klein) and a bottom-up connection (cf., Dreher et al., 2018). I describe this distinction as one dimension of mathematical connections–the dimension of *directionality*.

Dreher et al. (2018) give descriptions of top-down and bottom-up connections that exist in the literature. I summarize briefly. A *top-down mathematical connection* is *one that takes university mathematics as the starting point; it begins with the university mathematics (UM) and shows how such concepts or practices can be reduced, decompressed, or related to school mathematics (SM).* As one example, the authors point out that the field of real numbers can be constructed from the rational numbers in several ways (i.e., starting from university mathematics), but that discussion of this by means of nested intervals–as opposed to topological closure, Cauchy sequences, or Dedekind cuts–would be more suitable for school mathematics students. That is, we can infer a mathematical connection between UM and SM–namely, between the construction of $\mathbb{R}$ from $\mathbb{Q}$ (including several variants), and the notion of nested intervals that is evident in school mathematics largely via decimal representation. The key point is that if we take UM as the starting point–i.e., knowledge of various sophisticated ways to construct $\mathbb{R}$ from $\mathbb{Q}$–one might ask teacher candidates how this university mathematics can be connected to school mathematics concepts in a top-down connection.

In contrast to this, a *bottom-up mathematical connection* is *one that takes school mathematics as the starting point; it begins with school mathematics (SM) and shows how the concepts or practices of school mathematics are rooted in the structures of the discipline evident in university mathematics (UM).* To exemplify, the authors give an example pointing out that double paper-folding instructions sometimes used in school geometry courses (i.e., starting from SM), which produce perpendicular lines, is based on a particular definition of perpendicularity; namely, two lines g, h, are called perpendicular if $g \neq h$ and a reflection across h maps g onto itself. (A key point is that perpendicularity, in this double paper-folding, is not characterized by, for example, the dot product of two vectors being zero.) Here, the school mathematics topic has been taken as the starting point–double paper-folding–and teacher candidates can be asked how this SM topic reflects and might connect to UM concepts in a bottom-up connection.

Figure 1 depicts these two examples in terms of the directionality dimension of mathematical connections–top-down or bottom-up, which is represented by the directional arrows.

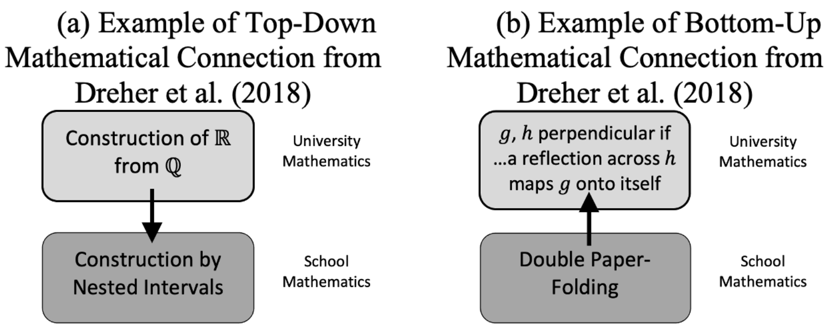

Figure 1. Examples of (a) top-down and (b) bottom-up mathematical connections
Dreher et al. (2018)

Didactical Connections: Top-down and Bottom-up directions

Fundamentally different from mathematical connections are didactical ones–that is, *a relation between mathematical concepts or practices encountered in university mathematics (UM) and the didactical approaches one takes to situations encountered in the activities of teaching school mathematics (TSM)*. Going beyond just mathematical connection-making, didactical connections are ones that inform instructional approaches. In short, it is a link between the mathematical and didactical planes; in particular, one in which university mathematics is connected to aspects of teaching school mathematics. In a manner similar to what was evident from mathematical connections, the literature appears to identify the same directionality dimension of these didactical connections–both top-down and bottom-up connections.

Stylianides and Stylianides (2014), for instance, talk about mathematics for teaching as a sort of "applied" mathematics. This idea represents a *top-down didactical connection*, which is *one that takes university mathematics as the starting point; it begins with the university mathematics (UM) and shows how such concepts or practices can be applied to resolve a situation encountered in teaching school mathematics (TSM)*. That is, it is a top-down directional arrow (like in Figure 1a), but between university mathematics and teaching school mathematics. In the authors' elaboration, prospective teachers first work "on a *mathematical* idea from an *adult*'s standpoint" (italics added, p. 272), and then a "pedagogical context [in which] prospective teachers [have] to consider…their mathematical work… from a teacher's standpoint" (p. 272). The point is that the adult mathematical concepts (i.e., UM) had some implication on the pedagogical context of teaching school mathematics, and were considered from this direction. As a particular example of such a top-down didactical connection, Wasserman and Weber (2017) explored how proofs of the algebraic limit theorems for sequences (in a real analysis course) could be used to inform a teacher's response to school mathematics situations about the use of rounded numbers while solving basic equations.

On the other side, Heid et al. (2015) considered all the mathematical concepts–including more advanced university mathematical concepts–that might relate to particular school mathematics teaching situations. This represents a *bottom-up didactical connection*, which is *one that takes situations encountered in teaching school mathematics as the starting point; it begins with the school teaching situation (TSM) and shows how university mathematical (UM) concepts or practices might be relevant to and arise from these school mathematical situations*. As an example, the authors started from a student questioning that $a^0 = 1$ for all nonzero real values of a, based on the student's explanation that a^0 would mean a times itself 0 times and so a^0 must be 0; they went on to explore, using the formal $\varepsilon - \delta$ definition of continuity, why defining $a^0 = 1$ makes sense because it means $f(x) = a^x$ is continuous on all real numbers. Here, the direction of the connection is in reverse (like in Figure 1b); it started with a pedagogical context of teaching school mathematics (TSM), but which then allowed for further explorations connected to some related university mathematics content (UM).

The literature base aiming to be more explicit about such didactical connections (and not just mathematical ones) has been more recent–in part, due to the development of, and alignment with, practice-based approaches to teacher knowledge and teacher education (e.g., Ball & Forzani, 2009). For example, a number of recent projects in the United States have created instructional modules for university courses such as real analysis, abstract algebra, modern geometry, and so forth, that try to elaborate such didactical connections (e.g., Álvarez et al., 2020; Lischka et al., 2020; Wasserman et al., 2019).

Another Dimension to Mathematical Connections

The distinctions above–between mathematical and didactical connections, and between top-down and bottom-up connections–situate some of the ways extant literature has attempted to capture various types of connections in order to counter Klein's second discontinuity. I described the top-down and bottom-up directions as the directionality dimension of such connections. Notably, this dimension was present in both mathematical and didactical connections. This dimension of connections is useful in that it provides a sense of what is foregrounded–that is, what is being presumed and from where the connection begins. In what follows I try to capture another dimension to mathematical (but not didactical) connections–what I refer to as the *set-relational* dimension.

Mathematical Connections: Set-relations

The set-relational dimension of mathematical connections considers the set relation between the two mathematical ideas in question–one from school mathematics (SM) and one from university mathematics (UM). In reality, the set-relation dimension describes a particular "framing" of the relationship between two mathematical ideas–for mathematical ideas have many facets. For example, a binary operation might be framed in ways that highlight the "binary" aspect–e.g., understanding it as a contrast to a "unary" operation–or in ways that highlight its "functional" aspect, etc. Set theoretic notions suggest there are four possibilities for such a framing, depicted in Figure 2, with descriptive titles given from the perspective of university mathematics.

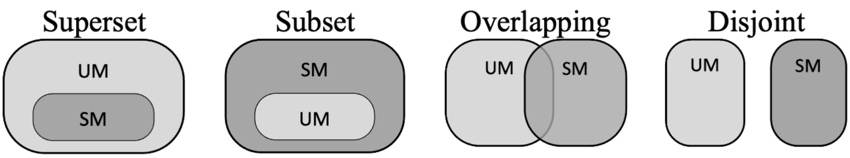

Figure 2. Four possible framings for set relations between university mathematics (UM) and school mathematics (SM)

Before I begin, what I intend to demonstrate is that the examples of mathematical connections described previously, regardless of directionality, appear to correspond to just one of the set relations–a superset connection. This fact suggests space for exploring this new set-relational dimension of mathematical connections. I then do so by considering the 'inversion' of this typical relation–a subset connection. Although I elaborate exclusively on superset and subset possibilities in this paper, future work might further consider overlapping and disjoint relations.

Consider the two examples given by Dreher et al. (2018) in Figure 1. Even though directionally they were different (as indicated by the arrows), if we consider each in terms of the mathematical concepts and relationships, what we see is that both represent a *superset relation*–one in which the *university mathematics (UM) concepts or practices are framed as a superset of the connected school mathematics (SM) concepts or practices*. In the universe of mathematical concepts, for example, we can consider the collection of those associated with the "construction of the set of real numbers, $\mathbb{R}$, from the set of rational numbers, $\mathbb{Q}$." Mathematically, there are many concepts that would fall within this notion, but, most germanely, there are also several different kinds of constructions. One of them is "construction by nested intervals." Through this framing, the concepts within this nested interval collection would not include, for example, Dedekind cuts, but it would include concepts such as sequences and decimal representations; in other words, construction by nested intervals is a subset of the broader collection. Notably, the superset is connected to UM; at university, we look at the various rigorous constructions (e.g., Dedekind cuts) and, for example, demonstrate that desirable properties are then preserved. The authors argue that construction by nested intervals is appropriate for SM because it addresses issues of the set of real numbers and of the field of real numbers; "school mathematics is essentially based on representations that facilitate an empirical inductive access using specific examples" (p. 334) and the denseness of the rational numbers in the reals is "illustrated by means of the decimal number representation of the rational numbers" (p. 334). Thus, the top-down mathematical connection described by Dreher is also an example of a superset connection. This set relation is depicted in Figure 3a. Similarly, Dreher et al.'s other example considers formal definitions of perpendicularity. In particular, we might consider the collection of concepts associated with "defining perpendicularity in terms of being when a reflection across *h* maps *g* onto itself." In UM, we might even consider other definitions of perpendicularity (such as dot products being zero), but for our purposes here, the key point is that the "double paper-folding" conducted in SM contexts exemplifies a particular one. The mathematical aspects of double paper-folding, again, represent a subset because this folding activity would be one way to exemplify the definition but there would be others. In sum, what we find is that, even though the connection is in the reverse direction, the fundamental set relation that is being framed between SM and UM is the same: they both represent examples where UM is a superset of SM (see Figure 3b). What this suggests is that the set-relational dimension is something distinct from the directionality dimension of mathematical connections.

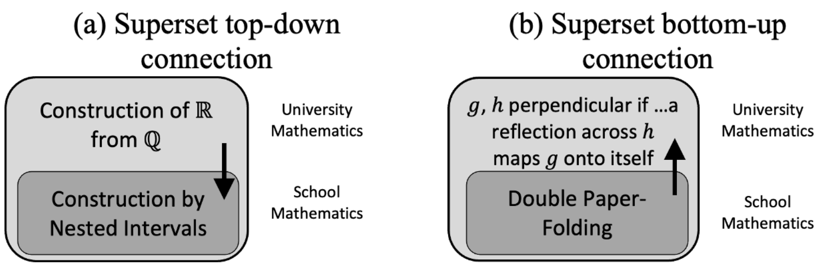

Figure 3. Dreher et al.'s (2018) (a) top-down and (b) bottom-up examples, as superset connections

Before moving on, the fact that both these examples represent a superset relation is not especially surprising. Essentially, as we continue further mathematical study, the mathematical concepts tend to get increasingly generalized and abstract–as noted previously. In SM we study a particular algebraic structure, whereas in UM we study algebraic structures more generally; in SM we study a particular geometry, whereas in UM we study various geometries; and so forth. The point is that in these very typical examples, SM is easily framed as a subset of UM; it is an example–often a more concrete one–of a broader and more abstract mathematical concept. Indeed, the primarily top-down mathematical connections from Klein's "elementary mathematics from a higher standpoint" tend toward this superset relational connection. (This superset connection is similar to what Wasserman and Galarza (2018) called a generalization connection.)

Exemplifying Subset Relational Connections

In contrast to the typical superset mathematical connections, let's consider an inversion of this relationship; namely, a *subset relation*, which is one in which *university mathematics (UM) concepts or practices are framed as a subset of the connected school mathematics (SM) concepts or practices*. From the outset, I want to clarify two things. First, that some of this is about conceptualization and framing–based on how SM was conceptualized earlier as well as the various framings one might take of UM concepts or practices. For instance, it is possible that even when we name a somewhat abstract topic in SM, such as "functions," we only associate the particular functions studied in school, which are typically functions of real variables (i.e., $\mathbb{R} \times \mathbb{R}$). Functions outside of these we might associate as being non-SM topics. However, based on the earlier description, if the SM concepts are being defined in a particular way, then they might still be used as instances in which it is reasonable to consider framing UM concepts or practices–or aspects of them–as a subset of this broader school notion. Second, that what constitutes SM and UM, of course, differs by context and country. The two examples I give below aim to be broadly applicable, but of course what I consider UM in these examples might be SM in some contexts and vice versa.

Although the functions studied in SM tend to be of a particular type (i.e., $\mathbb{R} \times \mathbb{R}$), the actual way function is defined can allow for abstraction–as discussed previously (e.g., CCSSM, 2010; Davis et al., 2010). In these contexts, the abstract definition of function appears to be part of SM. When we consider the abstract definition of function to be part of SM then the particular definition of, say, a binary operation in an abstract algebra course (UM) is an example of such a function (i.e., "A binary operation, *, on a set A, is a function, $*: A \times A \rightarrow A$). The point being, in this case, UM can be framed (at least in this aspect) as a subset of SM. That is, if the SM concept of function is "a relation where each element of the domain is associated with exactly one element in the range," then the following mapping which is an example of a binary operation is a function by that definition: $+:(x,y) \rightarrow x + y$ (for $x,y \in \mathbb{R}$). The point here is not that this example is or should be instantiated in SM, but rather that this framing provides an opportunity to make a mathematical connection between SM and UM–in a way that I argue is distinct from the other kind of connection which was a superset relation. I also argue that framing UM as a subset connection in this way shifts the learning; namely, the activity situates the SM concept of function as the broader focus of learning by its framing of UM as an example and thus places emphasis on the SM concept. (Wasserman (2023) and Wasserman & Galarza (2018) elaborated on use of this particular connection with secondary teachers; they also used the term instantiation connection to describe an idea similar to a subset connection.)

As another example, it is common in SM to justify area formulas using a "cut-reassemble" argument (e.g., cut the triangular end from a parallelogram and reassemble it into a rectangle). One way to conceptualize this sort of transformational argument is as an "area-preserving transformation"–meaning that such a transformation preserves a region's area. Indeed, even if this sort of argument in SM is informal and not given an abstract name, this sort of reasoning clearly exists in some form in this context. In this way, it is possible to situate "area-preserving transformations" as a relevant part of SM. Doing so allows us to frame other area-preserving transformations studied in UM as a subset connection. Wasserman et al. (2020) describe precisely such a connection by framing properties of the Riemann integral, such as $\int_a^b f - g = \int_a^b f - \int_a^b g$, as another example of an area-preserving transformation. Properties of the Riemann integral are typically framed in a university mathematics course as being about their "algebraic" properties, useful for understanding how we might operate with the integral concept. However, an alternative framing is to consider their meaning geometrically; that is, to frame the equality in the previous example as telling us something about the preservation of area (since the Riemann integral is frequently connected to area), and then abstracting the kind of transformation this property suggests. The property referenced above can be directly related to Cavalieri's principle, which at least in the U.S. is part of SM (CCSSM, 2010). Again, framing the connection between SM and UM in this way places the UM concept (properties of the integral) as another example of an area-preserving transformation (in addition to "cut-reassemble"), and thus emphasizes the SM concept of a transformation that preserves area as the focus of learning.

To reiterate the point that the directionality and the set-relational dimensions of mathematical connections are distinct, I use these two examples to give both top-down and bottom-up subset connections. According to its prior description, a top-down perspective starts from UM. So, we might start with the definition of a binary operation, and then ask about how one might use a functional mapping between sets to depict which elements are being mapped to and from in a particular binary operation (e.g., with the operation of addition, the pair (5,2) maps to the number 7). Because it begins with the UM it is top-down; because the abstract notion of function is part of SM and is the more general mathematical concept, it is a subset relation (see Figure 4a). Now let's consider the reverse. Suppose we start by pointing out that a "cut-reassemble" argument is an example of a transformation that preserves area (SM), and then ask how the integral property $\int_a^b f - g = \int_a^b f - \int_a^b g$ also exemplifies an area-preserving transformation. Because it begins with

SM it is bottom-up; because the integral property at university exemplifies this broader concept, it is a subset relation (Figure 4b). (Notably the descriptors top-down and bottom-up are counter to the direction of the arrows in Figure 4 because of where UM is situated in relation to SM.)

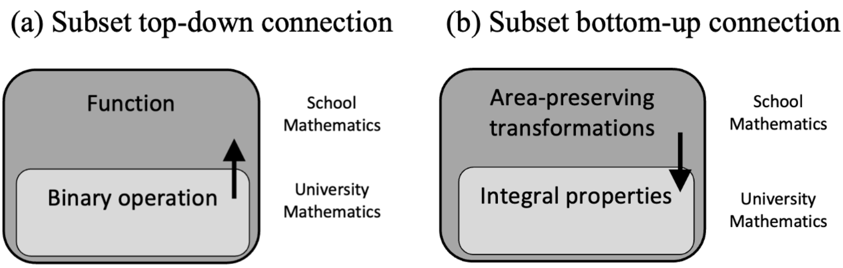

Figure 4. Examples of subset connections that are (a) top-down and (b) bottom-up

Discussion

Thus far, I have tried to articulate another dimension of mathematical connections between UM and SM–one that can be leveraged to help describe and differentiate various types of connections. Here, I draw out some summative ideas, and consider the rationale for discussing this new dimension–and the related subset mathematical connections–in terms of secondary teacher education.

An Expanded Framework of Mathematical Connections

When considering mathematical connections meant to bridge the gap between UM and SM, I have argued that current examples have tended to be differentiated along a dimension of *directionality*; in this paper, and up until this point, I have tried to argue that the *set-relational* facet is a distinct dimension that should also be considered. I will try to argue for why I regard such a distinction to be important in what follows, but for now it's helpful to summarize that in terms of *mathematical connections* (which are distinct from didactical connections), I argue we can differentiate along two different dimensions: directionality and set-relational. In the current discussion, I have only considered framings of superset and subset relations, although future work might further consider overlapping and disjoint relations. Table 1 provides a 2×2 table situating the now four different types of mathematical connections discussed and exemplified in this paper thus far (which are contained in Figures 3 and 4). A critical argument in this paper is that subset relational connections are less well-known, and less well-explored in the literature in relation to Klein's second discontinuity; indeed, given the natural progression of abstraction in mathematics, superset relations certainly seem more typical (and thus, subset relations potentially rarer).

Table 1. A 2×2 framework of mathematical connections between university and school mathematics

	Subset	Superset
Top-down	*Subset top-down* mathematical connections	*Superset top-down* mathematical connections
Bottom-up	*Subset bottom-up* mathematical connections	*Superset bottom-up* mathematical connections

Two Classes of Subset Mathematical Connections

Subset mathematical connections have in common the framing of UM as a subset of SM. Here, I discuss what I think are two general classes of this kind of mathematical connection; the purpose in doing so is that these classes might become starting points for generating more of these types of connections to counter Klein's second discontinuity.

Mathematical Practices

In mathematics education, it is reasonably common to differentiate between mathematical concepts and mathematical practices (NRC, 2001; MOE, 2020). Many mathematical connections–indeed, the majority of Klein's texts–focus on connections between mathematical concepts; say, between the addition of integers in SM and groups in UM. However, we can also consider connections between mathematical practices. Here, by mathematical practices, I am referring to activities such as problem-solving, proving, defining, generalizing, conjecturing, theoremizing, and so forth; the distinction being made is a contrast between particular mathematical concepts and particularly mathematical ways of engaging with those concepts, sometimes called mathematical "habits of mind" (Cuoco et al., 1996; Heid & Wilson, 2015; Mason et al., 2010; Rasmussen et al., 2005). Now, as mentioned previously, there are certainly differences between the mathematical practices at university and at school. That is, these mathematical practices undergo a didactical transposition within these institutional contexts (e.g., Ouvrier-Buffet, 2015). Proof is more heavily emphasized in UM than in SM, and has additional levels of rigor. Tall (1991), for instance, characterizes SM in terms of "describing" and "convincing," which in UM turn into "defining" and "proving" (in a logical manner based on those definitions). While differences certainly exist, there are also inherent commonalities across these mathematical practices–made more or less clear depending on the level of grain size at which they are described. What I argue here is that mathematical practices–as a class of mathematical connections–can often be conceptualized and framed as subset connections.

As Tall (1991) points out, there is a difference in what a definition looks and feels like at school and at university; in SM, they tend to be more descriptive, whereas in UM they are more stipulative. Nonetheless, defining is a mathematical activity in both places. That is, despite some of the differences, the university mathematical practice of defining can be framed in terms of the commonality it shares with the school mathematical practice of defining–a core kernel of the practice of defining. As such, I think it is possible to conceptualize this core of "defining" as a mathematical activity of SM, and seek to frame examples of defining in UM as particular examples of this broader mathematical practice. In doing so, we can conceptualize a subset mathematical connection between UM and SM. For instance, instead of framing the $\varepsilon - \delta$ definition of continuity in an analysis course as the starting point to discuss the concept of continuous functions, we might use it as an opportunity to discuss the activity of defining more generally, instead. We might ask students in a university mathematics course, for example, how they might define a continuous function; we might use those to interrogate how various definitions classify differently the same examples; and we might consider the advantages or disadvantages of certain characterizations. The key point is that, by framing the particular UM concept as an example of a broader mathematical practice–one whose core is also evident in SM–it is possible that students might gain an improved understanding of that core practice more generally. While not all of the specificities of how defining is practiced in SM versus UM will transfer, the broader practice provides a concrete point of connection between SM and UM–and, in particular, as a subset connection. As another example, Wasserman et al. (2019) describe "attention to scope" as a point of connection between the proofs of various derivative rules in a real analysis class, and broader considerations in teaching about providing explanations about school mathematics concepts; notably, this subset framing of the connection was valued by the prospective and practicing teachers in the study.

Furthermore, as a particular subset of mathematical practices, Wasserman (2022) conceptualized "pedagogical mathematical practices" (PMPs) as being at the intersection of mathematical practices and pedagogical practices for teaching mathematics. There being a genuine difference between these two spaces of mathematical practice and didactical practice in mathematics–as Weber et al. (2020) argue–PMPs, as a construct, asks the field to consider which ones are especially similar. Indeed, Wasserman and McGuffey (2021) report on the ways secondary teachers seemed to adopt and incorporate PMPs into their own pedagogical practice, after having had mathematical experiences with them in a real analysis course; the key point being that mathematical practices, and PMPs in particular, appeared to serve as a meaningful mathematical connection–a subset connection–that helped bridge Klein's second discontinuity.

Mathematical Structures

The definition of subset relational connections stipulates that concepts or practices in SM be framed as a superset of concepts or practices in UM. Although this may seem to invert the normative developmental progression in mathematics, broad mathematical structures might be another class of subset connections. Structures here is intended to convey an idea similar to Bruner's (1960) description "…to sense the simpler structure that underlies a range of instances…" (p. 68). A structuralist perspective on curriculum development, such as Bruner's, suggests that one should orient mathematics instruction around these broader structures, with complexity slowly and increasingly being layered on for further development (Howson et al., 1981). In this sense, the concept of "function"–from the earlier example–is an example of a large structure in mathematics (indeed, its strong emphasis in mathematics curriculum today is in part attributable to Klein!); an area-preserving transformation–or some other property-preserving transformation–similarly might represent another broader mathematical structure.

Critically, by highlighting and naming the broader structures which are encompassed in SM, we can frequently identify UM ideas as being examples or subsets of those structures. Doing so allows for making a subset connection. In addition to the function structural example earlier, we could also consider "equivalence" as a broader structure of SM. Through this lens, the study of quotient groups in UM might be framed not so much as another example of a group, but more used as an opportunity to explore its structural notion of equivalence–by regrouping collections into subcollections of objects which will be considered to be equivalent. In this sense, we are not "building on" equivalence as discussed in SM to link it with how equivalence is understood in UM, but rather we are acknowledging a core notion of equivalence as part of SM and framing the UM content as an opportunity to deepen one's notion of this core mathematical structure by exploring the UM as an instantiation of it. The key point is that by situating the particular UM idea as an example of a broader mathematical structure–one that is also tangibly present in SM, like equivalence–one gains an improved understanding of that structure more generally.

The key idea is that *mathematical practices* and broad *mathematical structures* are classes of connections that can represent subset mathematical connections; instances when ideas in UM can be conceptualized and framed as subsets of ideas relevant to SM. Both might be used to diversify and expand the kinds of connections typically made in university mathematics courses to counter Klein's second discontinuity.

Significance and Implications

As I have noted, many mathematical connections discussed in extant literature to counter Klein's second discontinuity between UM and SM are superset relations–where SM is an example of a broader idea in UM (cf. Dreher et al., 2018). Although these might differ in terms of directionality, whether top-down, i.e., "elementary mathematics from a higher standpoint", or bottom-up, i.e., "higher mathematics from an elementary standpoint," the set-relational dimension of mathematical connections is a distinct issue. Now, I argue not only that this dimension of connections is distinct, but that subset connections–as conceptualized in this paper–may be especially important for countering Klein's second discontinuity.

The key components to my argument for the value of subset relational connections with respect to Klein's second discontinuity has to do with (i) enriching SM conceptions instead of UM conceptions, by (ii) adding UM examples to one's example space for SM concepts, which might involve a reorganization of SM concepts, since (iii) doing so makes clearer the reference to when in the teaching of SM that a teacher might draw on this mathematical connection.

Consider the topic of a binary operation in abstract algebra. We might make various connections between this UM topic and SM concepts. We'll consider two possibilities–depicted in Figure 5. On one hand, we might consider a connection to function composition–$g \circ f$. In secondary school, students learn about function composition, primarily via algebraic substitution, where $(g \circ f)(x) = g(f(x))$. While discussing the general notion of a binary operation on a set, we might use function composition (or another familiar operation like addition) to exemplify this abstract concept. This is a superset connection. We have made a connection between UM and SM, and regardless of the directionality of the connection, note that a primary effect is to add a familiar and concrete example (from SM) to the example space of this new abstract idea of binary operation (in UM). That is, in Skemp's terms, we have expanded, or perhaps even restructured, the cognitive structures for the concept of binary operation. In this superset connection, what has been developed further–at least explicitly–is the idea from UM. I point out that such connections may be less effective at bridging Klein's second discontinuity because it is the UM idea (not the SM idea) that is explicitly foregrounded. On the other hand, as a different point of connection, we might make the connection to function described previously–by framing that a binary operation is an example of a function. This is a subset connection. And here, note that a primary effect is to add something concrete (from UM) to the example space of function–which is the SM topic. In Skemp's terms, adding this example expands, or even restructures, the cognitive structures for the concept of function. A subset mathematical connection like this explicitly foregrounds and further develops the SM concept;

meaning such subset connections primarily enrich SM conceptions, whereas superset connections primarily enrich UM conceptions. And since it is those SM concepts–like functions–that teachers will be responsible for teaching (not binary operations), having an enriched conception of a school topic arguably makes it more likely for teachers to recognize at what point in the curriculum those connections might become useful. While planning a lesson on function, for example, having an enriched conception of this SM topic is what acts as an explicit signal to a school teacher to consider the connection to UM.

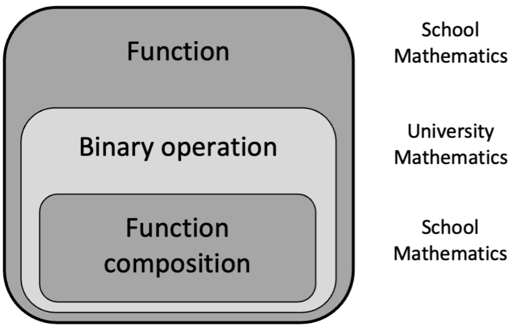

Figure 5. Possible superset and subset mathematical connections for binary operation

In addition to such theoretical arguments, which are based on conceptual differences between superset and subset connections, I also draw on my own work in the context of a real analysis course to support the potential value of subset connections. One module focused on the mathematical practice of "attention to scope," in which we used a sequence of proofs for the power rule for derivatives to exemplify this practice. What we found, reported on in Wasserman et al. (2019), was that this mathematical practice–indeed, a 'pedagogical mathematical practice'–was one that the teachers not only reported as valuable, but also incorporated into their own secondary teaching. That is to say, the subset mathematical connection made (bottom-up, in this case), which situated a sequence of real analysis proofs as an instance of "attention to scope," was one that appeared to be effective in terms of bridging Klein's second discontinuity; of helping clarify how prospective and practicing teachers' university studies could be related to the tasks of teaching school mathematics. Furthermore, as reported on in Wasserman and McGuffey (2021), we also found that teachers were attributing some of their approaches to teaching in relation to the strictly mathematical experiences in the real analysis course–and not just the modules that explicitly tried to connect to teaching. Although not reported on in either, participants in the course tended to find added value in those modules that leveraged subset connections; many were mathematical practice connections like "attention to scope," but others included more content-focused subset connections like area-preserving transformations. The key point is that, in addition to the theoretical arguments about their potential, some of my own work with teachers has indicated the potential value of using subset relational connections. Regardless, these indicate the potential of these subset mathematical connections to counter Klein's second discontinuity, with a call for further research, and demonstrate the value of articulating this additional dimension of mathematical connections.

Lastly, in terms of teacher education, the implications of the ideas in this paper are primarily for teacher educators. The framework presented is less important for teacher candidates to dwell on, and more useful for university teacher educators (including mathematicians who teach UM mathematics courses), who can use it to identify a wider variety of possible connections. The paper's contribution is that subset connections represent another type of connection–another tool in the toolkit, so to speak–for teacher educators to try to counter the second discontinuity Klein described. These depend on how one conceptualizes SM and how one frames UM; specifically, although UM concepts are interesting to study in their own right, by framing them as examples of broader school mathematical concepts, practices, or structures, one can find novel, interesting, and compelling connections between UM and SM that explicitly foreground and further develop SM.

Conclusion

There were two particular challenges for this special issue. The first was to highlight for university students the links between university mathematics and school mathematics; the second was to provide future teachers with access to effective tools for their didactic work. In regard to both of these challenges, this paper highlights a new *class* of links between UM and SM–specifically subset connections, which arose from elaborating the set-relational dimension of mathematical connections. While the particular examples in this paper may be useful, perhaps more useful is the theoretical identification and description of the class of subset connections; it is this theoretical classification (see Table 1), alongside the two classes of mathematical practices and broad mathematical structures, that allow for the creation of many more concrete examples. In this regard, it is the diversification of types and dimensions of connections to counter Klein's second discontinuity that becomes an effective tool for the didactic work of mathematics teachers–and mathematics teacher educators.

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