# A comparative study of the teaching of quadratic equations in five curricula: Brazil, France, Japan, Spain and Vietnam


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Comparative studies often aim to identify and explain differences and similarities of didactic phenomena in different school contexts. In mathematics education, comparative studies have been undertaken with a variety of aims, about a diversity of mathematics topics and their results depend on the theoretical approach, methods, and levels at which the comparison is done. We start by presenting the theoretical and methodological bases, according to the anthropological theory of the didactic (ATD), based on which we have carried out a comparative study about a specific mathematical theme in the context of 5 different educational systems. More concretely, this paper compares the institutionally offered curricula of five countries (Brazil, France, Spain, Japan and Vietnam) concerning the algebraic resolution of quadratic equations. This is achieved through an analysis of the curricula and a selection of representative textbooks. The objective is to identify the common and alternative didactic choices and the possible reasons for these choices, as well as the set of conditions and constraints set up in the different school contexts when this piece of knowledge is planned to be taught.



**Keywords**: quadratic equations, comparative analysis, curriculum analysis, textbook analysis, scale of didactic codeterminacy, reference praxeological model, dominant praxeological model, potentially offered curriculum



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## Estudio comparativo de la enseñanza de las ecuaciones cuadráticas en cinco planes de estudios: Brasil, Francia, Japón, España y Vietnam

Los estudios comparativos suelen tener como objetivo identificar y explicar diferencias y similitudes de ciertos fenómenos didácticos en distintos contextos escolares. En el ámbito de la enseñanza de las matemáticas, se han realizado estudios comparativos con objetivos muy diversos, sobre una diversidad de temas matemáticos y sus resultados dependen del enfoque teórico, de los métodos y de los niveles en los que se realiza dicha comparación. Empezamos presentando los fundamentos teóricos y metodológicos derivados de la teoría antropológica de lo didáctico (TAD), que nos han permitido realizar un estudio comparativo sobre un tema matemático especifico en el contexto de cinco sistemas educativos diferentes. En particular, este artículo se centra en la comparación de los currículos ofrecidos institucionalmente por cinco países (Brasil, Francia, España, Japón y Vietnam) en relación con la solución algebraica de ecuaciones cuadráticas, mediante el análisis de currículos y una selección de libros de texto representativos de cada país. Nuestro objetivo es identificar las decisiones didácticas comunes o alternativas y las posibles razones de estas decisiones, así como el conjunto de condiciones y restricciones que se establecen en los diferentes contextos escolares cuando se planea este objeto para ser enseñando.

**Palabras claves**: ecuaciones cuadráticas, análisis comparativo, análisis curricular, análisis de libros de texto, escala de codeterminación didáctica, modelo praxeológico de referencia, modelo praxeológico dominante, currículos potencialmente ofrecidos

## Une étude comparative de l'enseignement des équations quadratiques dans cinq *curricula* scolaires : le Brésil, la France, le Japon, l'Espagne et le Vietnam

Les études comparatives visent souvent à identifier et à expliquer les différences et les similitudes des phénomènes didactiques dans différents contextes scolaires. Dans le domaine de l'enseignement des mathématiques, des études comparatives ont été entreprises avec une variété d'objectifs, sur une diversité de sujets mathématiques et leurs résultats dépendent de l'approche théorique, des méthodes et des niveaux auxquels la comparaison est effectuée. Nous commençons par présenter les bases théoriques et méthodologiques issues de la théorie anthropologique du didactique (TAD), qui nous ont permis mener une étude comparative sur un thème mathématique spécifique dans le contexte de cinq systèmes éducatifs différents. Plus concrètement, cet article se centre sur la comparaison des *curricula* institutionnellement offerts par cinq pays (Brésil, France, Espagne, Japon et Vietnam) concernant la résolution algébrique d'équations quadratiques, à travers l'analyse des *curricula* et une sélection de manuels représentatifs. Nous cherchons à identifier les choix didactiques communs et alternatifs et les raisons possibles de ces choix, ainsi que l'ensemble des conditions et des contraintes mises en place dans les différents contextes scolaires lorsque cet objet de connaissance est prescrit pour être enseigné.

**Mots-clés** : équations quadratiques, analyse comparative, analyse d'un curriculum, analyse des manuels, échelle de codétermination didactique, modèle praxéologique de référence, modèle praxéologique dominant, curriculum potentiellement offert

# Introduction

International comparative assessments of student performance in mathematics such as the Trends in International Mathematics and Science Study (TIMSS) and the Programme for International Student Assessment (PISA) have motivated several research studies on the comparison of educational systems in different countries to explain differences in students' performance (Dindyal, 2020; Jablonka & al., 2018; Wu, 2006; Zhu & Fan, 2006; Son & al., 2017). As explained by Artigue and Winsløw (2010, p. 48), "Even if the measurement of student performance is not its central preoccupation, didactic research is certainly concerned with these debates". Our article is in line with this perspective and with a special focus on the research methods used for this comparative study with a didactic approach. These methods are strongly linked to the research questions we propose to address. Thus, in their research on the contributions of comparative studies of two education systems, France and Vietnam, Bessot and Comiti (2013) addressed two questions: What is relevant to compare? What can this comparison reveal to us? The authors rely on the analysis of ten PhD works carried out in the framework of an inter-university cooperation agreement between the two countries. One of the common aims of these works is to reinvest the results of the research community in didactics of mathematics while testing them in another context. More precisely, these comparative studies were considered, according to the authors, as a privileged way to question our context that we often consider as natural –a phenomenon that Chevallard (1999) describes as the "naturalization of taught mathematics"–. Complementarily, one might consider the specificities of our societies (e.g. linguistic, cultural) and of our educational institutions (organisation of the educational system, links between school and society, among others) when conducting comparative studies. As Artigue and Winsløw (2010, p. 49) point out: "It is clear that a didactical theory that could help inform and organise comparative studies would have to take a less naïve viewpoint on cultures and institutions".

For these reasons, and in the same line as stated by these authors, we base our research on the framework of the Anthropological Theory of the Didactic (ATD) (Chevallard, 1992a, 2019) by using some central of its theoretical and methodological tools. In this paper, we have chosen a specific mathematical domain, that of algebra, and within this domain, the specific piece of knowledge, that of solving quadratic equations. According to Hong and Choi (2014), algebra is one of the main domains and is recommended to be taught by all students worldwide. In particular, the topic of resolution of quadratic equations appears proposed for similar age groups (13–17 years) in different countries. The purpose of our research is to compare the *offered curricula by 5 countries concerning the algebraic resolution of quadratic equations to identify common and alternative choices when this piece of knowledge is planned to be taught and the possible raisons d'être of its introduction in educational systems.*

Our research is part of a more general phenomenon related to the didactic transposition of mathematical knowledge (Chevallard, 1985), which is presented in section 2. We focus on presenting a comparative study of 5 countries: Brazil, France, Japan, Spain and Vietnam. All of them with different teaching cultures and traditions. Section 3 presents the specific methodology used for this international comparison. The methodological choice of considering a reference praxeological model, related to the resolution of quadratic equation for the analysis of the knowledge to be taught makes it possible to carry out the comparative analysis among the 5 countries. Finally, in the last section, we present and discuss some of the main results of this comparative analysis that emerged from the cases here considered.

## I. Conditions for the existence of an object to be taught

To study the conditions for the existence of an "object of knowledge", we consider the existence of different institutions involved in the delimitation of an "object to be taught" (Chevallard, 1985; Chevallard & Bosch, 2014) in educational systems. An object to be taught is therefore a construct resulting from a process of transformations, most often beginning in the institution that produced it –what Chevallard (1985) characterizes as the didactic transposition process–. Consequently, there is a diversity of possible constructions of an object to be taught and of the curricula in which it fits (Artigue, 2018; Wijayanti & Bosch, 2018).

In our paper, we choose the official instructions and textbooks as observables to get access to the potentially offered curriculum (POC) for the educational system (secondary school education) (Chevallard, 2021). They are observables of different stages of curricula transformation and, consequently, of the different stages of the delimitation of the knowledge to be taught in the complex process of didactic transposition (Chevallard, 1985). The POC, on which we focus our paper, can resemble the intended curriculum, as called by Travers (1992), more recently adapted by Mullis and Martin (2015) for the TIMSS assessment framework. As explained in Barquero and al. (2023), according to the theory of didactic transposition, the analysis of curricula reforms requires taking into account a diversity of institutions (and agents who occupy different institutional positions) for its (re-)definition and implementation. In our comparative study, it is the POC produced by the institutions in charge of the knowledge to be taught.

A national curriculum exists in these 5 countries. These curricula delimit, in different or similar ways, the knowledge to be taught with explicit (or more often, implicit) epistemological and pedagogical prescriptions.

We consider the textbooks as especially pertinent as they represent an existing "agreement" between what can exist in the classroom, for students and for teachers. In the same sense described by Ravel (2002, p. 7, our translation):

> A textbook can be considered as a certain didactic preparation of the text of knowledge because it results from mathematical and didactic choices made by its authors on a given knowledge to be taught. However, this preparation does not take place within the same system of constraints as that made by the teachers [...].

This POC is the result of a set of conditions and constraints, which belong to different levels (i.e., societal decisions on curricula reforms, (dis)appearance of certain school subjects). To analyse these conditions realizable, we use the scale of didactic codeterminacy, introduced by Chevallard (2002), helping us to locate where conditions or constraints affect the existence and evolution of mathematical activity (see Figure 1). That is, if they appear at the more generic levels (including conditions set up by our society, schools, or pedagogies organising the way disciplines are taught and learned); or, at the more specific levels referring to the way that mathematical knowledge (going now into a discipline) is organised into different domains, sector, themes and tasks. In this paper, we use the different levels of the scale to present examples of conditions (also constraints) that exist in the different educational systems and that facilitate (or hinder) the existence of the object to be taught (in our case, of quadratic equations and its resolution). As explained by Chevallard (2002):

> Each level imposes, at a given moment in the life of the education system, a set of constraints and levers: the resulting ecology is determined both by what the constraints hider or push forward and by the exploitation that the actors will make of the levers that the different levels offer them (*Op. cit.*, p. 49, our translation).

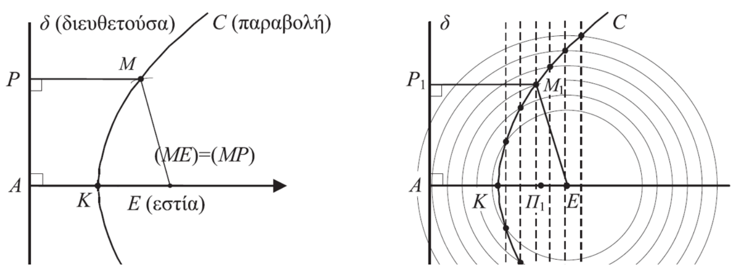



**Figure 1.** – The scale of didactic codeterminacy (adaptation from Chevallard, 2002)

*How to describe the POC in each country with the aim of comparing them?* Several studies (such as Bosch & Gascón, 2006) show that, in any institution, there is an implicit model of the mathematical knowledge to be taught. In other words, any didactic transposition of a piece of knowledge in a teaching system implicitly establishes a dominant model. These dominant models are the result of a set of conditions and constraints that allow the existence of certain practices in the institution and prevent others from appearing. As explained by Bosch and Gascón (2006, p. 57):

> Research in didactics needs to elaborate its own models of reference to be able to avoid being subject to the different institutions observed, especially the 'dominant' ones. There is no privileged reference system for the analysis of the different bodies of knowledge of each step of the didactic transposition process [...].

Particularly, when addressing a comparative analysis of what exists (and does not) in different educational systems, it is especially important to build and adopt a common reference model.

This reference model might be as much detached from any of the institutions into consideration, minimizing the risk of addressing empirical facts in a biased manner. We named this common model the reference praxeological model (RPM) which allows us to analyse the dominant models that exist in each particular educational system. As underlined by Bosch (2015, p. 61), the epistemological viewpoint adopted by researchers is always an *a priori* research assumption constantly evolving and continuously questioned. The reference epistemological models also determined the amplitude of the mathematical field where research problems are set out and the didactic phenomena which aims to be addressed by researchers

*Why do we talk about a praxeological model?* In the ATD, all human activity, including mathematical activity and its teaching and learning, can be modelled in terms of praxeologies (Chevallard, 1999). A praxeology consists of a quadruplet $[T/\tau/\theta/\Theta]$, a type of tasks T that can be accomplished by a technique $\tau$, justified by a technology $\theta$, which is itself legitimised by some theory $\Theta$. A praxeology is thus made up of two indissociable blocks, the block of "know-how" or *praxis* with $[T/\tau]$ and the "knowledge" block with $[\theta/\Theta]$. In the ATD, the reference epistemological models are formulated in terms of praxeologies and of sequences of linked praxeologies.

Based on these notions, the research questions addressed in this paper can be formulated more precisely in the following terms:

**RQ1**: What praxeological model of reference around the resolution of quadratic equations is necessary to access the dominant models of each educational system and analyse them?

**RQ2**: What conditions and constraints can explain the differences and similarities between the dominant models around quadratic equations in each educational system?

**RQ3**: What are the *raisons d'être*[1] that each dominant model imposes on the object to be taught, quadratic equations?

To produce elements of response to these research questions, we have developed a specific methodology that is explained in the following section.

# II. Methodology

The research methodology consists of three main steps.

The **first step** is guided by an exploratory inquiry into the conditions and constraints for the teaching of quadratic equations, from the *societal-school* levels to the *pedagogical-discipline* levels, and below, that are set up by the different educational systems of each country: Brazil, France, Japan, Spain and Vietnam. The selected observables are the national official instructions and textbooks of the 5 countries, which are initially analysed through the elaboration of a common questionnaire. Section III.1 presents examples of some common conditions recognised in the educational systems and the elaboration of a survey to get access to the particular ones.

The **second step** conducts an *a priori* analysis of the conditions of (co)existence of the praxeologies related to the *type of tasks* (T) about the algebraic resolution of quadratic equations from an epistemological point of view, that is without considering either the cognitive activity or the possible educational institutions involved. This first *a priori* analysis is considered as the reference praxeological model related to T and used to guide the analysis of observables of the *dominant praxeological* models in the different educational contexts. Section III.2 presents the reference praxeological model related to T.

This analysis is followed by the **third step** then focused on the description of the *existing praxeologies* around T by the selection of some representative textbooks in each country. With this purpose, in section III.3, we present a common grid to be used for the textbook's analysis. This grid is the result of enriching the first *a priori* analysis by considering what exists in each educational system.

1. "For every praxeology or praxeological ingredient chosen to be taught, the new epistemology should in the first place make clear that this ingredient is in no way a given, or a pure echo of something out there, but a purposeful human construct. And it should consequently bring to the fore what its raisons d'être are, that is, what its reasons are to be here, in front of us, waiting to be studied, mastered, and rightly utilised for the purpose it was created to serve. These are two necessary conditions for the diffusion of praxeologies to be meaningful." (Chevallard 2006, p. 31)

Table 1 summarizes the three steps followed, the main levels of the scale of didactic codeterminacy that are observed, the focus, and the specific methodological tools introduced to analyse the corresponding empirical data.

**Table 1.** – Synthesis of the steps, specific methodological tools and empirical data

| Section | Levels observed | Focus | Specific theoretical-methodological tools | Empirical data |
|---|---|---|---|---|
| III.1 | *Society-school level* | General regulation for official instructions and textbooks | Survey about official instructions and textbook policy | Official instructions and textbooks |
| | *Pedagogy-discipline level* | School organisation for the teaching of quadratic equations | | Official instructions and textbooks<br>Textbook (table of contents) |
| III.2 | *Subject* | *Reference praxeological model* around T | Analyse *a priori* of T[2] | Exploration of existing techniques related to T |
| III.3 | *Subject* | Enrichment of the *a priori* analysis from *existing praxeologies* | Grid for textbook analysis[3] | *Existing praxeologies* in a selection of textbooks |

This methodology has been implemented through regular videoconference meetings that facilitated in-depth discussions to reach a consensus on common English terminology, the interpretation of tools used, and analytical approaches. This organizational structure has enabled collaboration among researchers from various countries with diverse languages and cultures, all of whom share the same theoretical framework, the ATD.

## II.1. Inquiry into the conditions and constraints set up by educational systems

We seek to collect data that provide observables to investigate from the most *generic* conditions and constraints (*society-school* level) in the teaching and learning of quadratic equations at secondary school to the most specific levels of pedagogy and mathematical

2. The *a priori* analyses are supported by some central theoretical tools, such as the notions of pinpoint praxeologies, pivotal tasks, praxeological dynamics, pragmatic scope of techniques, cost of techniques, and techniques competition. These notions will be described later in the section III.2.

3. The construction of the grid is supported by the already introduced tools for the *a priori* analysis (see III.2) and considering the institutional scope of techniques existing in the selected textbook.

discipline (*domains-sectors-themes*, see Figure 1). The main empirical observables that have been chosen, for each educational system, are (a) the national (or regional, if it exists) curriculum and mathematics programmes in secondary school education, (b) the regulation concerning the publication and distribution of textbooks, and (c) a selection of the most representative textbook(s).

To have an approach to the conditions set up in each education system, we work on a common survey to be answered by each researcher, representative of each country authoring this paper. This survey is composed of two sections.

The *first section* focuses on describing the conditions offered by the *society* and the *school* system (level of the *society-school*). By conditions at these levels, we might mention a common one to the 5 educational systems about the dominant paradigm of *visiting works* (Chevallard, 2015) in what concerns the "work" or the "monument" of quadratic equations and their resolution.

> In the framework of the anthropological theory of the didactic, this paradigm is known as the paradigm of "visiting works" or—according to a metaphor used in ATD—"of visiting monuments", for each of those pieces of knowledge—e.g., Heron's formula for the area of a triangle—is approached as a monument that stands on its own, that students are expected to admire and enjoy, even when they know next to nothing about its raisons d'être, now or in the past. (*Op. Cit.*, p. 175).

As stressed by Chevallard (2002, p. 54, our translation):

> [...] A first point of view is that still dominant today of the system of *disciplines* (or knowledge, subjects, etc.) reduced to themselves as totalities, and not considered for what they would allow us in terms of knowledge and action: such is the *monumental approach* to knowledge and works. The School thus tends to appear as a more or less formal *cultural obligation* towards a few works whose choice is experienced by many as more or less arbitrary, and which must be visited in a hurry, without it being appropriate to linger too long.

The *school* level concerns a set of conditions and constraints that are related to the school institution itself. In the case of quadratic equations, schools in the 5 educational systems converge on considering that this piece of knowledge might be, with no doubt, included in the training of students at lower secondary school. Some differences in the grades where its teaching is planned have been detected, as will be explained in section IV.1.

Questions about the role and way curriculum is prescribed at the national level, the policy of publishing and distributing textbooks and the freedom of teachers to use these textbooks, were posed in this *first section*.

The *second section* aims to inquire into how the school organise the teaching and learning of *mathematics* (levels of the *pedagogy-discipline-domain-sector-theme*).

The *pedagogical* level frames the conditions offered in the study of *any* discipline. According to Chevallard (2002, p. 52, our translation), constraints imposed at the pedagogical level can be about the school disciplines that are delimited or the decisions on the short time for classroom sessions[4]:

> With no doubt we may refer here, [...] a constraint which is in no way disciplinary, but which is imposed *a priori* on all the taught subjects: [...] the fact that the study of any subject is squeezed into the short time of the one-hour session is generally opposed to the existence of didactic forms in which the work of the class on a given subject is articulated over a "long" period [...].

We included questions –concerning textbooks and official instructions about mathematics in different countries– to inquire into the more particular conditions and constraints at the levels of the *pedagogy* and the level of the *discipline* (and the more specific levels in *domains-sectors-theme*) planning the teaching of quadratic equations. The planned duration (i.e., in grades) for quadratic equations is an important condition to take into account, as well as where exactly are quadratic equations placed and how it is articulated to other possible themes.

Sections VI.1 and VI.2 present the results of the main important differences in the 5 educational systems about both sections of the survey.

Complementary, when we focus on analysing the conditions and constraints appearing at the most *specific* level, that is, the level of the *subject*, we aim to describe the praxeologies around the resolution of quadratic equations, as they exist in each of the 5 countries here considered. To make the comparative analysis realizable at this level of specificity, we need to consider a common *reference praxeological model* (presented in section III.2) that, later, allows us to define a common methodological tool to look in depth at the existing praxeologies. This tool takes the form of a grid presented in section III.3.

## II.2. An *a priori* analysis of the praxeologies related to the type of tasks T

The teaching of elementary algebra has been extensively explored within the framework of the ATD. Bosch (2015) provides an overview of various studies that have contributed to analysing the evolution of the didactic transposition processes that has led to a dispersion of the content traditionally assigned to "elementary algebra" in secondary school curricula. (*Op. Cit.*, p. 55).

In our paper, our aim is to analyse the variability in the existing praxeologies related a specific type of task: the resolution of quadratic equation. To facilitate the comparative

4. An important condition which is set up differently in each of the countries, is the time duration of the classroom sessions. In our context, for instance, in Brazil, France, Japan and Spain all the sessions are limited to 50 min.

study among the participating countries, the reference praxeological model focuses on the pinpoint and local praxeologies that can exist around the type of tasks T: "Solve in *R* a quadratic equation", which is here presented.

Our object of study can be placed in the domain of "algebra" or the one of "analysis", depending on if we refer to the current division of the school mathematics discipline. In the domain of analysis, it can be associated with quadratic functions and the approximate solution of a quadratic equation using an algorithm for finding the zeros of a function, such as that of Newton. Closer to the current curricula, this type of tasks is placed in the domain of algebra with the algebraic solution of quadratic equations. Our object of study is, therefore, the type of tasks noted T "Solve algebraically in *R* a quadratic equation"[5].

Around a single type of tasks, Chevallard (2019, p. 94) describes a pinpoint praxeology:

> A praxeology $\wp = [T/\tau/\theta/\Theta]$ is called a pinpoint praxeology, because it focuses on a "point," to wit the particular type of tasks T —one may say that $\wp$ is "built around" T.

According to this definition, a pinpoint praxeology can concern several techniques $\tau_i$ for the same type of tasks T. Chaachoua and Bessot (2019) introduce the notion of *elementary* pinpoint praxeology:

> For each technique performing the same type of tasks T, we define an *elementary pinpoint praxeology*: there are as many elementary pinpoint praxeologies as there are techniques performing a single type of tasks T. (*Op. cit.*, p. 236, our translation)

In this paper, we adopt the approach assumed by T4TEL[6] (Chaachoua, 2020) which describes a technique by a sequence of types of tasks. The different techniques considered are those inventoried during the exploration of existing techniques related to T. Each of the techniques has a specific technological and theoretical element. Table 2 synthesizes the elementary pinpoint praxeologies around T.

5. We may highlight that T is deliberately limited to allow a comparative analysis, being also realistic with the place where the resolution of quadratic equations is placed in the different educational systems.

6. T4 refers to the praxeological quadruplet (Task, Technique, Technology, Theory) and TEL to Technology Enhanced Learning. The original motivation for developing T4TEL was to formalise the praxeological elements and their relationships for a computer representation. For example, a type of tasks is represented by a pair (verb, complement), a technique as a sequence of types of tasks (for more details, Chaachoua (2020)). This development finds its interest within the ATD outside the context of computer developments as shown in this paper.

**Table 2.** – Elementary pinpoint praxeologies of T

| Technique $\tau$ | Description of $\tau$ as sequence of tasks | Technology linked to $\tau$ |
|---|---|---|
| $\tau_{nul\_product}$ | • $T_{nul\_regroup}$ (Group all terms into one)<br>• $T_{factorise}$ (Factorise an algebraic expression)<br>• $T_{nul\_product}$ (Solve an equation of the form $P \times Q = 0$) | Null product property<br>Properties of conservation of equalities<br>Distributive property<br>Remarkable identities[7] |
| $\tau_{square\_root}$ | • $T_{regroup\_cte}$ (Group all terms with x in one side and constants in the other)<br>• $T_{factorise}$ (Factorise an algebraic expression)<br>• $T_{squared\_cte}$ (Solve an equation in the form $(ax + b)^2 = k$) | Square root rule<br>Conservation properties of equalities<br>Distributive property<br>Remarkable identities |
| $\tau_{sp\_evident\_root}$ | • $T_{trinomial}$ (Put in the form $ax^2+bx+c = 0, a \neq 0$)<br>• $T_{evident\_root\_eq}$ (Find by trial and error a solution to $ax^2+bx+c = 0$)<br>• $T_{sum\_or\_product}$ (Write the sum or product relation of roots)<br>• $T_{1st\_degree\_equation}$ (Solve a first-degree equation) | Relationship between roots and coefficients of the trinomial<br>Properties of conservation of equalities<br>Arithmetic properties of numbers |
| $\tau_{sp\_trial\_root}$ | • $T_{trinomial}$ (Put in the form $ax^2+bx+c = 0$, $a \neq 0$)<br>• $T_{sum\_and\_product}$ (Write the sum and product relation of roots)<br>• $T_{trialerror\_root\_syst}$ (Find the solutions of a system of first-degree equations by trial and error) | Relationship between roots and coefficients of the trinomial<br>Properties of conservation of equalities<br>Arithmetic properties of numbers |
| $\tau_{discriminant}$ | • $T_{trinomial}$ (Put in the form $ax^2+bx+c = 0$, $a \neq 0$)<br>• $T_{discriminant}$ (Apply the discriminant formula) | Discriminant formula<br>Conservation properties of equalities<br>Distributive property<br>Remarkable identities |

7. These three equalities are called 'remarkable identities' in France and Spain. These are: (1) $a^2 - b^2 = (a - b)(a + b)$; (2) $a^2 + 2ab + b^2 = (a + b)^2$; (3) $a^2 - 2ab + b^2 = (a - b)^2$. In Japan, these same three equalities are expressed 'expansion' or 'factorisation' formula' depending on the position of the terms. For instance, the equality (2) that transforms $a^2 + 2ab + b^2 = (a + b)^2$ is called 'factoring expression' but the expression (2') $(a + b)^2 = a^2 + 2ab + b^2$ is called 'expanding expression' (Keirinkan, Math 3 for Junior High School, Chapter 1. Expanding and factoring expressions). The choice of terminology was collectively made to enhance communication among researchers.

The description of the five techniques in Table 2 involves three particular types of tasks:

- $T_{nul\_product}$ (Solve an equation in the form $(ax + b)(cx+d) = 0, ac \neq 0$)
- $T_{squared\_cte}$ (Solve an equation in the form $(ax + b)^2 = k, a \neq 0$)
- $T_{trinome}$ (Put in the form of a trinomial $ax^2+bx+c = 0, a \neq 0$)[8]

These three types of tasks are sub-types of tasks generated from T by the variable (Chaachoua & Bessot, 2019) "form" of the quadratic equation. The decision to introduce, in the description of the techniques, these three sub-types of tasks are linked to the technological environments that justify and discriminate them. That is the reason why, we call these sub-types of tasks *pivotal tasks*. For example, the technological element "Null Product Property" necessarily relies on the form $(ax + b)(cx+d) = 0$ of T. Each of these pivotal tasks emphasises a particular form which we denote by *factorise form* by $(ax + b)(cx+d)$, *square-constant form* by $(ax + b)^2 = k$ or *trinomial form* by $ax^2 + bx + c = 0$.

Once we have introduced the first step of *a priori* analysis, we continue by studying the conditions of existence and coexistence of the techniques and thus of the elementary pinpoint praxeologies. For this reason, we introduce the notion of the pragmatic scope of techniques[9], which is defined, according to Kaspary, Chaachoua and Bessot (2020, p. 247) as follows:

> The *pragmatic scope of a technique* is the set of tasks where the technique is reliable in the sense that it can accomplish these tasks with little risk of failure and at a reasonable cost. The technique tends to succeed within this scope and tends to fail outside it.

The three sub-types of tasks distinguished above $T_{nul\_product}$, $T_{squared\_cte}$, $T_{trinome}$ are included respectively in the pragmatic scopes of the technique $\tau_{nul\_product}$, $\tau_{square\_root,}$ and $\tau_{discriminant}$. But, even if they are pivotal tasks, they do not completely characterize them. Indeed, the notion of pragmatic scope is delicate to be delimited. To better understand this interplay, let us consider theoretically what can happen in the case of the *coexistence* of several techniques to accomplish the same type of tasks. We then talk about *competition between techniques* and the *cost of a technique*, inspired by the Theory of Didactic Situations (Brousseau, 1998).

Competition between techniques suggests that one may be more efficient than the other on a certain set of tasks in the sense that it is less costly than its competitor. For example, we judge $\tau_{square\_root}$ to be less costly than $\tau_{discriminant}$ or $\tau_{null\_product}$ for the task *t* "solve algebrai-

8. A technique for solving Ttrinomial (Put in the form $ax^2+bx+c = 0$, $a \neq 0$) can be: (1) Put all terms in one member, (2) Expand an algebraic expression, and (3) Reduce an algebraic expression. This form is thus obtained by expansion and reduction.

9. The notion of pragmatic scope corresponds to Chevallard's definition (1999).

cally the equation $x^2 - 7 = 0$". This optimality of $\tau_{square\_root}$ on this competition domain leads to the belonging of $t$ to the pragmatic scope $\tau_{square\_root}$ and the exclusion of $t$ with respect to the pragmatic scopes of $\tau_{discriminant}$ and $\tau_{null\_product}$.

## II.3. Textbook analysis grid for the comparative study

In the *a priori* analysis (section III.2), we have identified (independently from any textbook and from any teaching institution) three pivotal forms: *factorise, square-constant* and *trinomial* — associated respectively with the three sub-types of tasks $T_{nul\_product}$, $T_{squared\_cte}$, $T_{trinome}$ —. However, there are many other forms, in particular those taken by the starting equation. One of the questions of our study is to identify the institutional values of the variable forms and the ecological reasons for these values. For this purpose, we have listed in a grid (see Figure 2, column D) the tasks included in the textbooks of each country. Then, researchers associated each task found in the textbook with one sub-types of tasks $T_i$. These sub-task types provide information about institutional choices.

This division into *sub-types of tasks* $T_i$ was therefore constructed by going back and forth between, on the one hand, the *a priori* analysis centred on the sub-types of pivotal tasks and, on the other hand, the assignation of the list of tasks in the textbook to their sub-types of tasks. However, the expected techniques for each $T_i$ may differ from one institution to another and may not be optimal from an epistemological point of view (pragmatic scope). To take into account the institutional specificity of each of the 5 education systems, we use the notion of *institutional scope* (Kaspary *et al.*, 2020).

> The institutional scope of a technique for a type of tasks T is the set of tasks where this technique is expected by an institution. This scope is a consequence of the conditions and constraints of the life of $\tau$ in an institution.

The *institutional* scope and *chronogenetic* aspects—that is, the didactic time how the introduction of the different sub-type of tasks and techniques (and other praxeological elements) by each manual contributed to enriching the institutional description of the sub-types of tasks $T_i$ related to T. For example, we have distinguished between the two sub-types of $n(k)^2x^2 = n(k')^2$ and $n(k)^2x^2 - n(k')^2 = 0$, while they belong to the same pragmatic scope of $\tau_{square\_root}$. In Japan, these two sub-tasks are within the institutional scope of $\tau_{square\_root}$ whereas in France only the first one is within the institutional scope of this technique, the second being within the institutional scope of $\tau_{nul\_product}$

**Table 3.** – Example of institutional description of sub-type of tasks $T_i$ related to T

| $T_i$ Country | $n(k)^2x^2 = n(k')^2$ | $n(k)^2x^2 - n(k')^2 = 0$ |
|---|---|---|
| Japan | Institutional scope of $\tau_{square\_root}$ | |
| France | Institutional scope of $\tau_{square\_root}$ | Institutional scope of $\tau_{nul\_product}$ |

Figure 2 shows an excerpt from the grid organising the data collection, as well as the process undertaken to collect the observables for a given textbook (column B) of a given school level (column A).

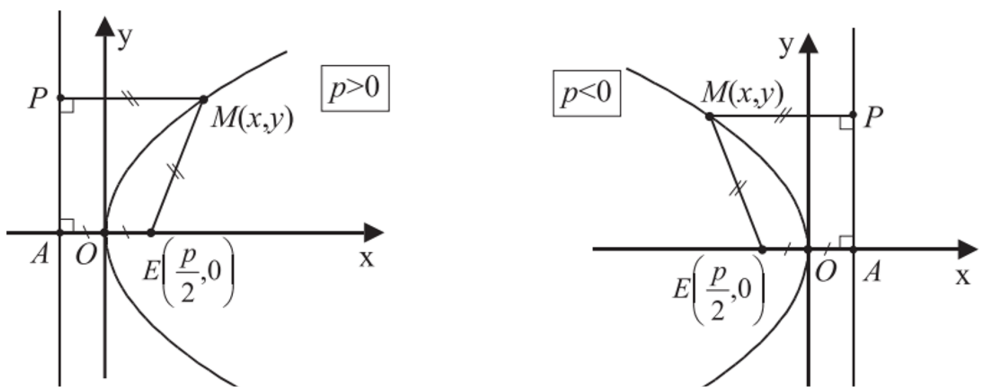



**Figure 2.** – Extract from the grid for the case of France

The empirical data selected from the textbooks (specified in column B) have been analysed following the procedure, common to all the researchers, and synthesised in the same grid used in the 5 country contexts.

(i) We have analysed chapters of the textbook where T is proposed as a teaching object. All (sub)tasks from T are identified and copied into column D (see Figure 2). We have not included tasks where the instruction is not explicitly "solve the equation", such as more open tasks leading to solving an equation.

(ii) For each task, its order of appearance (column C), its expression (column D), the nature of the roots (column H) and the pedagogical or didactic role of the task (column I) are included. Columns H and I are filled in using a drop-down menu (firstly agreed by the researchers).

(iii) Then, each task included in the textbook, is assigned to an *a priori* sub-type of tasks $T_i$ (column G). Complementarily, for each task, the expected technique by the institution is described, and analysed through the textbook proposal. It corresponds to the institutional scope (column F). And, the pragmatic scope anticipated in advance, based on the *a priori* analysis of type (and sub-types) of tasks and techniques associated (column E) presented in section IV.

To facilitate the analysis to the researchers, columns E, F and G are filled in using a drop-down menu. A key methodological element is the determination of the values of the drop-down menu in column G. The choice of the division of T into task sub-types $T_i$ was the result of a back-and-forth process between the analysis of the manuals of each country and the choice of the values of the form variable which characterise the sub-types of tasks. The breakdown was refined to reflect the ecological conditions and constraints of each institution. In particular, one of the criteria is that a sub-type of task (G) be included in the institutional scope of a single technique[10]. The values in the drop-down menus in columns E and F are the 5 techniques identified in the reference praxeological model about T.

# III. Results

We present the results of the three focuses of our research necessary to characterise some elements of the *dominant praxeological models* of the institutionally offered curriculum related to T in the 5 countries.

## III.1. Differences and similitudes at the society-school levels

For each of the 5 countries, Table 4 describes the school levels where quadratic equations, and their resolution, are proposed to be taught and learnt (shaded cells). Columns (3-7) include the school levels in each country and, the first two columns described the correspondence to the common coding and corresponding ages. The countries are grouped by geographical area: South America, Western Europe, and South-East Asia. One can appreciate the different lengths of study ranging from 1 to 3 school years. The earliest introduction is in Brazil and Spain (grade 8) and the latest end of study is in France (grade 11).

**Table 4.** – School levels and programme's duration for the study of $T_{\text{quadratic\_equations}}$

| Code | Ages | Brazil | France | Spain | Japan | Vietnam |
|---|---|---|---|---|---|---|
| Grade 8 | 13-14 | 8º | 4° | ESO2 | | |
| Grade 9 | 14-15 | 9º | 3° | ESO3 | 9th grade | 9 |
| Grade 10 | 15-16 | 1º | 2° | ESO4 | | |
| Grade 11 | 16-17 | 2º | 1° | Bachillerato1 | | |
| Grade 12 | 17-18 | 3º | Terminale | Bachillerato2 | | |

10. A technique for solving $T_{\text{trinomial}}$ (Put in the form $ax^2 + bx + c = 0, a \neq 0$) can be: (1) Put all terms in one member, (2) Expand an algebraic expression, and (3) Reduce an algebraic expression. This form is thus obtained by expansion and reduction.

For each country, as summarised in Table 5, we sought to indicate the specific features of each country: the existence of a national and regional curriculum, the way it prescribes teachers' practice and the policy of publishing and dissemination of textbooks.

**Table 5.** – Curriculum and Textbook Policy

| | | Brazil | France | Spain | Japan | Vietnam |
|---|---|---|---|---|---|---|
| **Curricula** | National curriculum | x | x | x | x | x |
| | Regional adaptation of curriculum | x | | x | | |
| | Teachers' curriculum adaptations | x | x | x | x | |
| **Textbooks** | Unique textbook in the country | | | | | x |
| | Several textbooks | x | x | x | x | |
| | External revision of textbooks | x | | | x | x |
| | Selection by the school/teacher | x | x | x | x | |
| | Free expenses of textbooks | x | | | x | |

In Vietnam, the existence of a single textbook published by the Ministry provides an official interpretation of the curricula and institutional expectations. For other countries, the multiplicity of textbooks expresses a form of pedagogical freedom and the possibility of the coexistence of different interpretations of curricula by the textbook authors. However, in Japan and Brazil, the textbooks are validated by the ministry for conformity with the national curriculum, through complex evaluation processes[11].

Furthermore, in all countries except Vietnam, teachers have the option of using the official class textbook as one of the resources for both the more theoretical and practical parts. In Vietnam, the teacher is obliged to follow strictly the organisation indicated by the textbook. However, the teacher is free to choose exercises from other resources.

For the rest, the empirical data we base our analysis on is, on the one hand, the curriculum of each country and, on the other hand, the selection of textbooks with some criteria for their selection, except for Vietnam where the unique textbook of grade 10 has been analysed. More concretely, for the cases of Brazil and Japan, the selected textbooks correspond to the ones validated by the Ministry and the most representative, that is, the ones produced by the best-selling editorial houses. For France, textbooks have been selected following the results of a survey distributed to 270 teachers to know about the most used ones. In the case of Spain, this selection has been made also according to the best-selling editorial houses and to the ones used by the schools the researcher used to collaborate with.

11. See for example the case of Brazil (Bittar, 2020).

## III.2. Differences and similitudes at the pedagogical-didactic levels

If we first focus on locating the quadratic equations in the national curriculum, in all the countries the teaching and learning of quadratic equations and the study of T is linked to the school domain of *algebra*. In those countries where at least two years are devoted to the study of T (as in Brazil, France and Spain) in the first year(s) T is more clearly linked to the domains of arithmetic and algebra. It is proposed through the resolution of equations in different numeral settings (*N, Z, Q, R*). In the last year, T is closely proposed to make the link between the resolution of equations, inequations, systems of equations and functions. In the cases of Vietnam and Japan, its teaching is proposed in one grade. It is strictly associated with the domain of algebra, focusing on the resolution of equations, with few links to other mathematical domains or sectors (such as the one on functions).

If we move to the analysis of the selected textbooks, all the textbooks are consistent with the divisions into domains and their reference to sectors, already introduced by the curriculum where T is located.

Table 6 summarises the editorial houses that were selected by each country, the grades corresponding to each of the textbooks, and the numbers and titles of the chapters that have been analysed.

**Table 6.** – Selected textbooks and corresponding grades and chapters analysed

| Country | Editorial houses | Grades | [Grade, Number of chapters (Ch) or sections (Sc)]<br>Title of chapters/Title of subsections |
|---|---|---|---|
| Brazil | FTD, "A conquista da Matemática" | 8 and 9 | [G8, 1Ch/1Sc] Equations/Equation of second degree<br>[G9, 1Ch] Equation of second degree |
| | FTD, "Matemática, Realidade & Tecnologia" | 8 and 9 | [G8, 1Ch/1Sc] Equation, system of equations and inequalities/ Equation of second degree witch an unknown<br>[G9, 1Ch/1Sc] Algebraic expressions and quadratic equation/ Equation of second degree witch an unknown |
| | Moderna | 8 and 9 | [G8, 3Ch] Powers and roots<br>Notable products and factoring<br>First degree equation systems with two unknowns<br>[G9, 1Ch] Equation of second degree |
| France | Myriade | 9 | [G9, 1Ch/1Sc] Equations and Inequations/Solve problems leading to first degree |
| | Indigo | 9 | [G9, 1Ch/1Sc] Equations and Inequations/Solve an equation |
| | Transmath | 9, 10 and 11 | [G9, 1Ch ] Use literal calculation to solve an equation<br>[G10, 1Ch/1Sc] Literal calculation/Application to solving equations<br>[G11, 1Ch/1Sc] Equations and polynomials functions of degree 2/Solving the second degree equation $ax^2 + bx + c = 0$ |
| | Maths'x | 10 and 11 | [G10, 2Ch/2Sc] Square and square roots/Square functions<br>Modelling by a function. Equations./Solve equations<br>[G11, 1Ch/1Sc] Polynomials of degree 2/Equations $ax^2 + bx + c = 0$ |
| | Indice | 10 and 11 | [G10, 1Ch/1Sc] Numerical calculation – Literal calculation/ Equations<br>[G11, 1Ch/1Sc] Second degree/Equation of second degree |
| Spain | Edebé | 8, 9 and 10 | [G8, 1Ch] Equations with 1 unknown<br>[G9, 1Ch] Quadratic equations<br>[G10, 1Ch] Equations and systems of equations |
| | Santillana | 9 and 10 | [G9, 1Ch] Equations of 1st and 2nd degree<br>[G10, 1Ch] Equations and inequations |
| Japan | Keirinkan | 9 | [G9, 1Ch] Quadratic equations |
| | Tokyo Shoseki | 9 | [G9, 1Ch] Quadratic equations |
| | Suken Shuppan | 9 | [G9, 1Ch] Quadratic equations |
| Vietnam | Ministry of Education edition | 9 | [G9, 1Ch/2Sc] Function $y = ax^2$, $a \neq 0$. Quadratic equations/ Equations of 1st and 2nd degree/Formula Quadratic equations |

This table allows us to get an idea of where quadratic equations are located and how textbooks name the parts of the book dedicated to their study. In most cases, the study of quadratic equations is carried out in one chapter. This reveals a didactic choice that consists of dedicating a separate time to the teaching and learning of this object.

The presence of the resolution of quadratic equations in sub-chapters is an indicator of the connection of this resolution to other mathematical objects, as in Vietnam with connection with functions (although they are weak and restricted to $y = ax^2$).

## III.3. Differences and similitudes at the subject level: Static analysis

In this part of the analysis, we place ourselves at the *specific* levels of didactic codeterminacy to analyse, through the selection of textbooks and specific chapters, the praxeologies proposed to be taught in each of the 5 countries. In all 5 educational contexts, we find the three elementary punctual praxeologies around the techniques $\tau_{nul\_product}$, $\tau_{square\_root}$, $\tau_{sp\_evident\_root}$ that have been identified in the *a priori* analysis.

This first part of the comparative analysis studies the *institutionally offered curriculum* in each of the selected textbooks without considering the different sequencing proposed for the study of $T_{quadratic\_equation}$. We aim to analyse the tasks proposed in the textbooks to guide the student's work. The *a priori* analysis of $T_{quadratic\_equation}$ has led to the distinction of 18 sub-types of tasks $T_i$, whose presence varies in each country (see Table 7, 2nd row with the number of $T_i$). The number of $T_i$ varies from a minimum of 6 sub-types of tasks in the case of Vietnam to a maximum of 18 in the case of France, with a range of 12.

For each country, we define the *ratio of the number* of tasks (*n*i) of a certain type $T_i$ over the total number *N* of tasks present in the textbook. If we define the ratio $ni / N$ as the *weight of a sub-type of task T*i, we can consider that a sub-type of tasks is significant in relation to the other sub-types of tasks $T_i$ *if its weight is greater than a certain threshold*. In our study, we have considered a threshold of 5%. If we look at the number of significant tasks (see Table 7, 3rd row), the minimum is 5 (in Vietnam) and the maximum is 8 (in Spain). Thus, the amplitude of the range of significant $T_i$ becomes 3. We might thus highlight that, in the selected textbooks for the analysis, there is a clear focus on a reduced number of types of tasks $T_i$ (having the rest underrepresented).

In this respect, two pedagogical strategies can be identified in the textbooks. The first consists of presenting a variety of types of sub-tasks $T_i$, among which only a small number are significant. France or Japan provides examples of adopting this first strategy. The second strategy consists of including only those $T_i$ that have a significant weight, as is the case of the Vietnamese textbook.

**Table 7.** – Number of $T_i$ in the textbooks in each of the five countries

| Country<br>Number of | Brazil | France | Spain | Japan | Vietnam |
|---|---|---|---|---|---|
| $T_i$ | 11 | 18 | 11 | 14 | 6 |
| Significant $T_i$ (at 5%) | 7 | 7 | 8 | 7 | 5 |
| Average numb. of the tasks associated to a significant $T_i$ (at 5%) | 27 | 20 | 37 | 48 | 11 |

The last row of Table 7 informs about the tasks average belonging to the same significant task $T_i$. We may comment on two extreme cases: Japan offers 48 tasks on average of the same $T_i$, while Vietnam only 11. The other three countries (Brazil, France and Spain) are more comparable in the average number of tasks. These differences make us infer different pedagogical strategies. For example, in the case of Japan, the pedagogical approach seems to be to make students work for a longer time on the same praxis (related to the same sub-type of tasks). When, in the case of France, fewer tasks are proposed (20 compared to 48) linked to the same sub-type of task $T_i$.

*What is the significant Ti for each country?* Knowing the nature of the $T_i$ is essential to studying the most dominant *praxis* to access the dominant praxeological model of each institution. At the 5% level, there are 11 significant sub-types of tasks $T_i$ indicated in the following Table 8 (see Appendix 1 for the complete list of sub-types of tasks).

**Table 8.** – Set of significant sub-types of tasks $T_i$ derived from the analysis of the five countries

| Sub-type of tasks | Generic expression of $T_i$ |
|---|---|
| $T_1$ | $ax^2 = c, ac > 0$ |
| $T_3$ | $[nk^2]x^2 - [nk'^2] = 0$ |
| $T_4$ | $ax^2 + c = 0; ac < 0$ |
| $T_6$ | $ax^2 + bx = 0$ |
| $T_7$ | $ax^2 + bx + c = 0$ |
| $T_8$ | $ax^2 + bx + c = 0; a \pm b \pm c = 0$ or $4a \pm 2b \pm c = 0$ or $9a \pm 3b \pm c = 0$ |
| $T_9$ | $[ka^2]x^2 \pm [2kac]x + [kc^2] = 0$ |
| $T_{10}$ | $k(ax + b)^2 = k'; kk' > 0$ |
| $T_{15}$ | $[nk^2](ax + b)^2 - [nk'^2](cx + d)^2 = 0$ |
| $T_{16}$ | $k(ax + b)(cx + d)(\ldots) = 0$ |
| $T_{20}$ | $P_2(x) = Q(x)$ |

Before discussing the relationship between significant $T_i$ and their associated techniques, we analyse the presence of techniques in the selected textbooks. Table 9 presents, for each country, the distribution of the presence of the 5 techniques $\tau_{nul\_product}$, $\tau_{square\_root}$,

$\tau_{sp\_evident\_root}$, $\tau_{sp\_trial\_root}$ and $\tau_{discriminant}$ (solving $T_{quadratic\_equation}$) in the textbooks. Table 9 indicates the techniques that are more represented in each country. They appear as the *dominant* techniques proposed in textbooks. The crossed-out cells indicate the absence or the *quasi* absence of a technique (with less than 5%).

**Table 9.** – Frequency distribution of the presence of *expected techniques* by country

| Technique / Country | $\tau_{nul\_product}$ | $\tau_{square\_root}$ | $\tau_{sp\_evident\_root}$ | $\tau_{sp\_trial\_root}$ | $\tau_{discriminant}$ |
|---|---|---|---|---|---|
| Brazil | 29% | 44% | 0% | 2% | 25% |
| France | 54% | 15% | 13% | 0% | 18% |
| Spain | 33% | 25% | 0% | 0% | 42% |
| Japan | 34% | 35% | 0% | 0% | 23% |
| Vietnam | 8% | 23% | 21% | 5% | 43% |

The two techniques $\tau_{square\_root}$ and $\tau_{discriminant}$ are present in all 5 countries, but in different frequencies:

- In Brazil and Japan, $\tau_{square\_root}$ is the dominant technique. In Japan, $\tau_{square\_root}$ coexists, in similar frequencies, with $\tau_{nul\_product}$ and $\tau_{discriminant}$, while the other techniques are almost absent. Let us notice a specificity of Japan about $\tau_{discriminant}$ in which we will focus. In grade 9, only equations with real number solutions are proposed. In the case of equations in the form of a trinomial and having two distinct solutions, the following "Formula for the solution of quadratic equations" is used (see Figure 3).

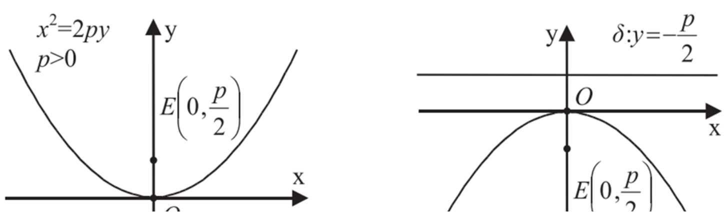



**Figure 3.** – Description of the "solution formula" (Keirinkan, Math 3, p. 67)

In addition, double roots can basically be solved by factorisation, so they usually do not appear in solution formula situations. The use of the following "Formula for the solution of quadratic equations" is avoided in the case of a double root in favour of $\tau_{nul\_product}$. In the case of Japan, and as shown in Okamoto *et al.* (2015b), it is only with the introduction of complex numbers in grade 10 that the discriminant is introduced and allows some practice with coefficients like with tasks:

> "If the quadratic $2x^2 + 3x + m = 0$ has two different real roots, find the range of values of the constant $m$."
>
> "If the quadratic equation $2x^2 + (m - 2)x + m + 4 = 0$ has a double root, find the value of the constant m and the double root."

The technique of $\tau_{discriminant}$ is the dominant one in the cases of Spain and Vietnam, with more than 40% of appearance. This technique is also relatively present in Brazil and Japan, with about 25%. It is in the case of France where only 18% of the frequency of appearance corresponds to the introduction and use of this technique.

About the other two represented techniques $\tau_{nul\_product}$ and $\tau_{sp\_evident\_root,}$ we may highlight significant differences. The $\tau_{nul\_product}$ technique, almost absent in Vietnam, is present in the other four countries: Brazil, France, Spain and Japan. It appears with similar frequencies in Brazil, Spain and Japan, and it is one dominant in France. $\tau_{sp\_evident\_root}$ is present in France and Vietnam, with a surprisingly high weight in Vietnam (26%).

To conclude this section, we want to mention some relationships between the 11 significant sub-types of tasks $T_i$ (listed in Table 8) and the three dominant techniques $\tau_{nul\_product}$, $\tau_{square\_root}$ and $\tau_{discriminant}$, to continue characterizing the dominant praxis in each country. Indeed, we hypothesize that the *dominant praxis* (at the 5% threshold) is a crucial element to characterise the dominant praxeological model. First, we question the significant sub-types of tasks $T_i$ that belong to the pragmatic scopes of the three dominant techniques. In other words, which ones of the significant T can be associated with the pragmatic scope of these dominant techniques? Table 10 summarises the distribution of the significant sub-types of tasks T to the corresponding dominant technique in the selected textbooks for each of the countries.

**Table 10.** – Correspondence of the significant $T_i$ to the pragmatic scope of the 3 dominant techniques in the textbook of the 5 countries

| | Brazil | France | Spain | Japan | Vietnam |
|---|---|---|---|---|---|
| $\tau_{nul\_product}$ | $T_{6,}$ $T_9$ | $T_6$ $T_{15}$, $T_{16}$ | $T_6$, $T_{16}$ | $T_6$, $T_{16}$ | $T_6$ |
| $\tau_{square\_root}$ | $T_1$, $T_3$, $T_9$ | $T_1$ | $T_1$, $T_{3,}$ $T_4$ | $T_4$, $T_{10}$ | $T_4$ |
| $\tau_{discriminant}$ | $T_7$, $T_{8,}$ $\mathbf{T_{20}}$ | $T_7$, $T_8$, $\mathbf{T_{20}}$ | $T_7$, $T_8$, $\mathbf{T_{20}}$ | $T_7$, $T_8$, $\mathbf{T_{20}}$ | $T_7$, $\mathbf{T_8}$, $T_{20}$ |

In this table, we have indicated in *bold* the significant sub-types of tasks $T_i$ face to which the institutional scope of a technique is different from its pragmatic one. Both sub-types of tasks $T_8$ and $T_{20}$ are concerned:

Concerning $T_{20}$ [$P_2(x) = Q(x)$] in the textbooks (except the case of Vietnam), other techniques than $\tau_{discriminant}$ are expected outside their pragmatic scope. In Brazil, France and Spain, $\tau_{nul\_product}$ is predictable. In Japan, at least in the textbooks selected for the analysis, $\tau_{nul\_product}$, the more prioritized technique is the $\tau_{square\_root}$, which is consistent with the high percentage of presence of this technique (corresponding to 35%, see Table 8).

For $T_8$ [$ax^2 + bx + c = 0$; $a \pm b \pm c = 0$ or $4a \pm 2b \pm c = 0$ or $9a \pm 3b \pm c = 0$], in the case of Vietnam, this sub-type of tasks is used to work with the $\tau_{sp\_evident\_root}$ technique (corresponding to a 21%, see Table 9).

It can be also noticed, on the one hand, that the *praxis* around $\tau_{discriminant}$ (or the variant in the case of Japan) are the same and common to any institutionally offered curriculum presented by the selected textbooks of the 5 countries: ($T_7$, $\tau_{discriminant}$), ($T_8$, $\tau_{discriminant}$), ($T_{20}$, $\tau_{discriminant}$).

On the other hand, the variability of the *dominant praxis* around the other techniques indicates a different study path not only due to the length of this path (from 1 year to 3 years) but also by the didactic conditions set up for the implementation of p*raxis* around $\tau_{discriminant}$. This leads us to question the *dynamics of the praxeologies proposed to be taught* in the selected textbooks of different countries, which is developed in the following section.

## III.4. Differences and similitudes at the subject level: Dynamics analysis

We enrich the previous analysis by considering the *chronogenesis* of the techniques leading to $\tau_{discriminant}$, as it informs about additional didactic conditions for its study and about possible competition between $\tau_{discriminant}$ with other techniques. We limit ourselves to the three main detected techniques $\tau_{nul\_product}$, $\tau_{square\_root}$ and $\tau_{discriminant}$.

### Chronogenesis of the techniques

We propose to examine how each country organises the introduction and study of different techniques. We are interested in looking at when the $\tau_{discriminant}$ technique is initially expected in the task sequence. Figure 4illustrates, for each country, the techniques worked before the appearance of $\tau_{discriminant}$ with the number of occurrences of praxis (Ti , τj) where τj is different from $\tau_{discriminant}$. Then, because of space limitation in the paper, we grouped in "Others" all the praxis worked after the first encounter with $\tau_{discriminant}$. The detail of the techniques grouped in "Others" is given in Appendix 2. In this figure[12], we show only the school level where the $\tau_{discriminant}$ technique appears for the first time (e.g., grade 8 in Spain, or grade 11 in France).

12. $P_2(x)$ denotes a polynomial of second degree, $Q(x)$ denotes a polynomial of degree 1 or 2.

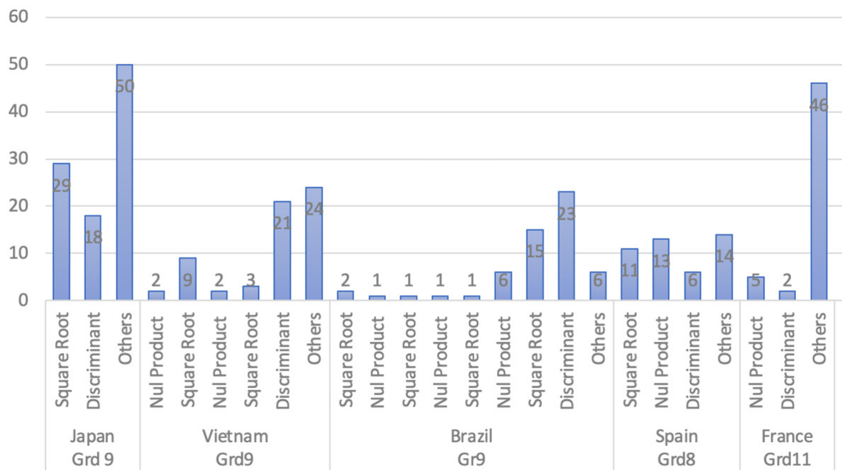



**Figure 4.** – Techniques worked before the appearance of $\tau_{\text{discriminant}}$
Legend: The ordinate axis represents the number of successive tasks where the technique $\tau_i$ is expected.

In the rest of the analysis, we also consider the work developed at the rest of the school levels to better understand the *chronogenetic* aspects of the techniques.

We first consider the countries that plan the study of T during 2 or 3 school years/grades. These are the cases of Brazil, France and Spain.

- In Brazil, in grade 8, in 2 over the 3 textbooks, the study of T is mainly focused and using $\tau_{\text{square\_root}}$ through the proposal of resolution of equations of the form "$ax^2 = b$" [corresponding to $T_1$] –or those that can easily be reduced to this form. In the third textbook, there appears a wider variety of equations (from grade 8 onwards) and the proposed work revolves around the technique $\tau_{\text{nul\_product}}$. In grade 9, in the three textbooks, the resolution of quadratic equations is more linked to the work of factorization. The techniques $\tau_{\text{nul\_product}}$ and $\tau_{\text{square\_root}}$ continue to be present and combined for the resolution of T. Then the technique $\tau_{\text{square\_root}}$ is consolidated before working on the new technique $\tau_{\text{discriminant}}$ intensively (40% of 56 tasks). Very few tasks are worked on afterwards. The $\tau_{\text{discriminant}}$ technique is seen as an outcome of the T.
- In France, during the first two years –grades 9 and 10–, the study of T is a pretext for the non-functional algebraic factorization work –in grades 9 and 10–, as well as its resolution linked to the study of functions –in grade 10 (antecedents in the study of quadratic functions). The technique of $\tau_{\text{discriminant}}$ is absent in these two grades. The institutional *raison d'être* of its late introduction in grade 11 is to overcome some limitations of the previous techniques and to be able to solve some types of tasks that other techniques could hardly succeed. The competition

of two main techniques $\tau_{nul\_product}$ and $\tau_{discriminant}$ is assumed and presented as a new sub-type of tasks, making explicit some technological-theoretical elements of the corresponding praxeologies. The three techniques $\tau_{square\_root}$, $\tau_{nul\_product}$ and $\tau_{discriminant}$ are worked on alternately without giving more weight to the $\tau_{discriminant}$ technique.

- In Spain, the formula of the discriminant appears since the first year (grade 8, in the 3 textbooks). The other techniques are introduced and worked alternately with the $\tau_{discriminant}$. In the rest of the sequencing, this same way of organising tasks, according to the appearance and alternating of techniques, is repeated by increasing the complexity of the tasks. Other techniques than $\tau_{discriminant}$ appear as economical palliatives of this main technique $\tau_{discriminant}$.

In the other two countries, Japan and Vietnam, the study of T is entirely developed in grade 9.

- In Japan, for selected textbooks, $\tau_{square\_root}$ is worked intensively to be consolidated before the consideration of $\tau_{discriminant}$. The work on the two techniques $\tau_{square\_root}$ and $\tau_{discriminant}$ at the beginning represents about 30% of the 167 tasks. Then, as in France, the three techniques $\tau_{quare\_root}$, $\tau_{nul\_product}$ and $\tau_{discriminant}$ (grouped in "Others") are worked alternately without giving more weight to $\tau_{discriminant}$.
- In Vietnam, after a first work on the two techniques $\tau_{square\_root}$ and $\tau_{nul\_product}$, intensive work on $\tau_{discriminant}$ is planned (with 30% of the 61 tasks). Any more kind of work in other techniques, such as $\tau_{sp\_evident\_root}$ and $\tau_{sp\_trial\_root}$, is planned.

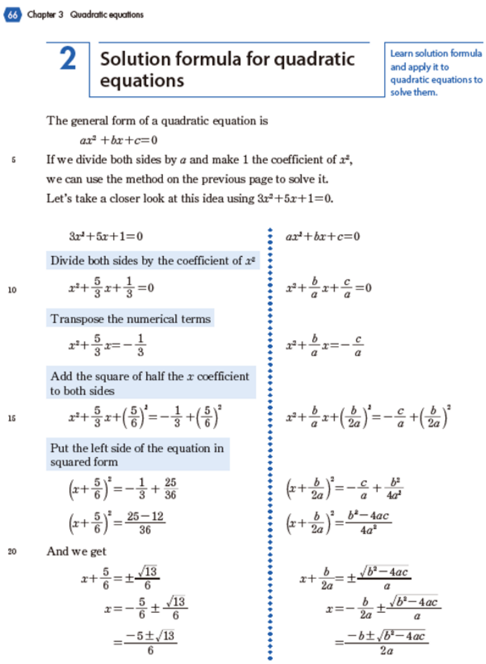
Based on the relation between the introduction and development of the different techniques (and the corresponding tasks and technological elements), we can infer the role that these techniques have in the production of two formulas (around the discriminant), which themselves contribute to the technological environment of $\tau_{discriminant}$.

We have identified in the textbooks of the 5 countries two technological environments for producing the discriminant formula.

On the one hand, the technology $\theta_{squared\_root}$ justifies $\tau_{square\_root.}$ and allows to reduce a trinomial into the type of pivotal task $T_{square\_cte}$ (cf. excerpt from Japanese textbook, Figure 5).

On the other hand, the technology $\theta_{nul\_product}$ justifies $\tau_{nul\_product}$ and consists of reducing the trinomial to the type of pivotal task $T_{nul\_product}$ (cf. excerpt from French textbook, Figure 6).

In both cases, the formula of the discriminant contributed to the emergence of a technique $\tau_{discriminant}$ coming to simplify and unify the resolutions, which makes easier the transformations of the starting equation towards one of the two types of pivotal tasks: $T_{squared\_cte}$ and $T_{nul\_product}$.



66 Chapter 3 Quadratic equations

## 2 Solution formula for quadratic equations

Learn solution formula and apply it to quadratic equations to solve them.

The general form of a quadratic equation is

$$ax^2+bx+c=0$$

If we divide both sides by $a$ and make 1 the coefficient of $x^2$, we can use the method on the previous page to solve it. Let's take a closer look at this idea using $3x^2+5x+1=0$.

| | |
|---|---|
| $3x^2+5x+1=0$ | $ax^2+bx+c=0$ |
| Divide both sides by the coefficient of $x^2$ | |
| $x^2+\frac{5}{3}x+\frac{1}{3}=0$ | $x^2+\frac{b}{a}x+\frac{c}{a}=0$ |
| Transpose the numerical terms | |
| $x^2+\frac{5}{3}x=-\frac{1}{3}$ | $x^2+\frac{b}{a}x=-\frac{c}{a}$ |
| Add the square of half the $x$ coefficient to both sides | |
| $x^2+\frac{5}{3}x+\left(\frac{5}{6}\right)^2=-\frac{1}{3}+\left(\frac{5}{6}\right)^2$ | $x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2$ |
| Put the left side of the equation in squared form | |
| $\left(x+\frac{5}{6}\right)^2=-\frac{1}{3}+\frac{25}{36}$ | $\left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$ |
| $\left(x+\frac{5}{6}\right)^2=\frac{25-12}{36}$ | $\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$ |
| And we get | |
| $x+\frac{5}{6}=\pm\frac{\sqrt{13}}{6}$ | $x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{a}$ |
| $x=-\frac{5}{6}\pm\frac{\sqrt{13}}{6}$ | $x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{a}$ |
| $=\frac{-5\pm\sqrt{13}}{6}$ | $=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ |

**Figure 5.** – Excerpt from the Japanese textbook about the "solution formula" (Okamoto *et al.*, 2015b, p. 66)

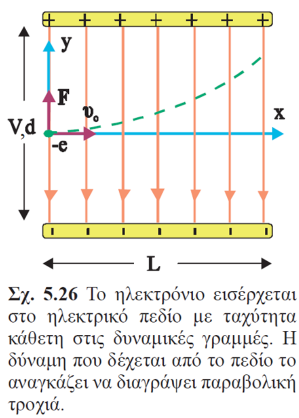



**Figure 6.** – Excerpt from the French textbook with the relation of discriminant and number of solutions (Dhombres *et al.*, 2019, p. 68)

### Conditions to allow this competition between techniques and $\tau_{\text{discriminant}}$

We examine the conditions offered by the textbooks for a potential competition of the $\tau_{\text{discriminant}}$ technique with the other two techniques $\tau_{\text{square\_root}}$ and $\tau_{\text{nul\_product}}$.

In the three countries Brazil, Spain and Vietnam, almost all the tasks dealt with are of the trinomial or binomial types of $T_i$. That is, the type of tasks $T_7$ or the sub-type of tasks ($T_1$, $T_3$, $T_4$, $T_7$ and $T_8$): 75% in Brazil, 90% in Vietnam and between 70% and 80% in Spain. More concretely, we consider the case of Brazil: for 30% of the tasks that are within the pragmatic scope of the technique $\tau_{\text{nul\_product}}$, the institutionally expected technique is $\tau_{\text{discriminant}}$. For example, this is the case for the sub-type of tasks $T_9$: $[ka^2]x^2 \pm [2kac]x + [kc^2] = 0$, which is in the pragmatic scope of the technique $\tau_{\text{nul\_product}}$ (see Appendix 1). Given the dominance of these types of tasks and the order in which techniques are introduced and work, it can be said that the textbooks do not seem to offer favourable conditions for a possible competition of alternative techniques with $\tau_{\text{discriminant}}$.

In these three countries, the lack of competition between $\tau_{\text{discriminant}}$ with the other two techniques $\tau_{\text{square\_root}}$ and $\tau_{\text{nul\_product}}$, leads to the situation that kinds of sub-tasks of T are assigned to the institutional scope of $\tau_{\text{discriminant}}$, whatever its cost.

In Japan, as in France, the variety of sub-types of tasks and the order in which the techniques are introduced and worked offer better conditions for possible competition among techniques (see Appendix 2). It is thus expected that the discussion around the cost of the techniques would be present when addressing T.

## Conclusions

In this paper, we have sought to account for the specificities of educational institutions (organisation of the educational system, links between school, society, culture, etc.) to characterize what conditions and constraints the teaching of a certain object of knowledge in different educational systems and therefore *in fine* what can finally happen inside the classroom. As Artigue and Winsløw (2010, p. 49) point out:

> It is clear that a didactical theory that could help inform and organise comparative studies would have to take a less naïve viewpoint on cultures and institutions.

To carry out the comparative analysis of the 5 educational systems, we started by considering a common *reference praxeological model* around the object of knowledge "Solve algebraically in *R* a quadratic equation" (**RQ1**). This type of task noted T is taught and learned in the 5 countries. The elaboration of the *reference praxeological model* (RPM) has been made in successive comings and goings of *a priori* analyses, first of the possible praxeologies around T, then of the praxeologies around *sub-types of tasks* $T_i$ of T. This *a priori* analysis has led us to question the epistemological conditions of (co)existence of these praxeologies.

For this purpose, we have considered the notions of *competition of techniques* to solve the same type of tasks and introduce the notions of *cost* and *pragmatic scope* of a technique.

This enrichment was only possible thanks to a constant dialectic between the considered reference praxeological model (at a certain moment) with the researchers' inquiry into different textbooks of each of the countries looking for what does and does not exist in terms of praxeologies. This is how the sub-types of tasks $T_i$ of T were better determined.

The first element of the RPM of possible praxeologies around T has been presented in Table 2. The elements enriching this initial RPM –derived from the common surveys and first inquiry into textbooks of the 5 countries– are summarized in Appendix 1. This table specifies the sub-types of tasks $T_i$ of T, more or less present and significant according to the countries, and the *epistemologically possible praxis* ($T_i$, $\tau_i$). Being $\tau_i$ one of the 5 techniques present in the 5 education systems, $T_i$ belongs to the pragmatic scope of $\tau_i$. Recall that if $T_i$ belongs to the pragmatic scope of $\tau_i$, $\tau_i$ is the optimal technique, in terms of cost, to solve $T_i$.

The RPM, which is a research hypothesis, made it possible to collect and question what exists in the dominant praxeological models of the 5 educational systems considered about T and how T is planned to be taught and learned in the different countries through the analysis of official instructions and textbooks.

To describe the conditions and constraints that can explain the differences and similarities between the dominant models around quadratic equations in each education system (**RQ2**), there appears another central tool which is the levels of didactic codeterminacy (Figure 1). This scale is used as a theoretical-methodological tool to place and discuss at which level certain conditions or constraints appear affecting T. To this end, our research focuses not only on the specific levels of how mathematics discipline is organised but also on the higher levels, i.e. beyond the didactic system.

If we start from the *specific* level, the *subject* level, the RPM has played a central role, as it has constituted the common lens on how to get access to and describe the dominant praxeological models in each country through the analysis of selected textbooks. We have to highlight two matching analyses: the static and the dynamics, which provide complementary information.

On the one hand, the *static analysis* has revealed some important conditions for the *institutionally offered curriculum* in each of the selected textbooks without considering the different sequencing proposed for the study of T. This analysis has exposed important phenomena around the number of sub-types of tasks and the kinds of techniques that are proposed to be taught concerning T. Regarding the number of sub-types of tasks that are more proposed, our analysis shows the reduced number of *significant tasks* (between 5 and 8) that the analysis in each educational system reveals. But, at the same time, different choices are made about which sub-types of tasks are more (or less) proposed. Concerning the *average number of tasks associated with a significant* $T_i$, there are important differences among the countries (from 11 to 48), which is closely related to the didactic decision of how different educational systems decide how to work on the selected $T_i$ (Table 7). About the techniques, the analysis brings light on the techniques that are more represented, the *dominant techniques* proposed in textbooks. The three most present techniques in the 5 countries are $\tau_{nul\_product}$, $\tau_{square\_root}$, $\tau_{discriminant}$ but there are differences in their frequency (Table 9). Moreover, we have presented the correspondence of the significant sub-types of tasks $T_i$ to the pragmatic scope of these 3 dominant techniques in the textbook of the 5 countries (Table 10). This has revealed some common traits of the *dominant praxis,* such as the dominance of $\tau_{discriminant}$ around the same sub-type of tasks. But also, some difference that seems to indicate a different study path not only due to the length of this path (from 1 year to 3 years) but also by the didactic conditions set up for the implementation of dominant praxis around $\tau_{discriminant}$.

This leads us to introduce the *dynamics of the praxeologies proposed to be taught* in the selected textbooks of different countries. The analysis aims to analyse the *chronogenetic* introduction and development (that is, considering the didactic time by considering the length of the school years and the order of how $T_i$ and $\tau_i$ are presented, among other aspects) of the most present techniques $\tau_{nul\_product}$, $\tau_{square\_root}$ and $\tau_{discriminant}$. Complementing the previous analysis, this dynamic analysis has made explicit some conditions that each country set up when they plan the study of different techniques. More concretely, we initially discuss the techniques that are worked before the appearance of $\tau_{discriminant}$ (Figure 6), with correspondence of the school grade. We then analysed the conditions that each country offers for the concurrence and competition between techniques, helping us to infer the role of the techniques in the production of the discriminant formulas (discriminant), which themselves contribute to the technological environment of $\tau_{discriminant}$.

If we move to the more generic levels, at the *pedagogical-disciplinary* level, a common condition in the 5 countries is that the study of T is linked to the domain of algebra. In France, Spain and Brazil, the study of T is spread over 2 or 3 years, but in Japan and Vietnam, it is concentrated in 1 year. Our study shows the presence of isolates or quasi-isolate praxeologies around T in the 5 educational systems. In some of the cases, T presents some links with the solution of equations, inequations, systems of equations, and functions. Moreover, the structure of textbooks, and the chapters and sections involving T (Table 5), facilitated getting an idea of where quadratic equations are located and the kind of terminology used to express the *domain-sector-theme* where T is placed. This also provides information about the didactic choices (or conditions) in each country about how to split (or concentrate) over the time the study of T.

Concerning the conditions at the *school-pedagogical* level, we have interrogated the 5 countries concerning the curriculum and textbook policies and the freedom of teachers to use textbooks. In the 5 countries, quadratic equations have been detected as a piece of knowledge defined in the curricula but with remarkable differences in the grades and duration where its teaching is planned. Regarding the textbook policies and teachers' use, although remarkable differences between the Asian and European countries, these conditions are important to be considered to understand the variability in the analysis in the lower levels of codeterminacy.

At the higher level of codeterminacy, the *society-school* level, the dominance of the *paradigm of visiting works* (Chevallard, 2015) seems to be undeniable and appears as a common condition (or an important and restrictive constraint) common to the 5 educational systems analysed. The visit of the work of "solving quadratic equations" conducted in the selected textbooks of the 5 educational systems invariably leads to a formula that explicitly or implicitly (Japan) involves the discriminant.

Our analysis reveals that the current *raison d'être* for solving quadratic equations (**RQ3**) seems to be common to all 5 countries: to produce a "formula" (the discriminant one) whose scope is optimal compared to other possible techniques. The *raison d'être* of the introduction of the techniques seems to be to prepare the technological environment for the introduction of the $\tau_{\text{discriminant}}$. Although some differences that our analysis has shown, in the lower levels of codeterminacy, all is thus influenced by these important constraints of visiting the work T, and more in particular, our investigation shows how students are driven towards the "visit" of the discriminant technique and of the discriminant formula.

In scholarly knowledge, an *important raison d'être* of solving quadratic equations is the factorization of polynomials. Depending on the country, this *raison d'être* appears to be more or less important. An inquiry with students' resolution of quadratic equations in the 5 countries (whose analyses are now in progress) shows the primacy of the use of the discriminant formula for solving quadratic equations, even for equations clearly outside the pragmatic scope of $\tau_{\text{discriminant}}$. For example, consider the results of students (observed data in France, grade 11, 2021) for $x^2 - 7 = 0$. Out of 49 students, 33 students used the discriminant to find the roots of this equation, with 26 of these students finding two correct roots.

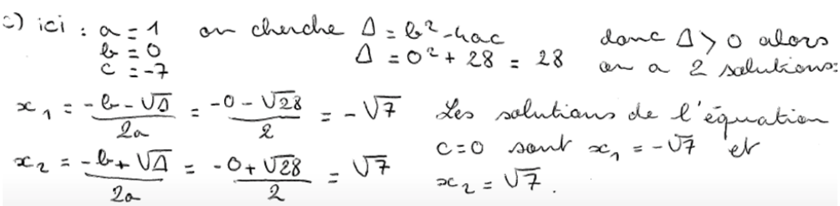



**Figure 7.** – Resolution of a French student of $x^2 - 7 = 0$

In our current study, we are conscious that we have limited ourselves to the knowledge to be taught related to T. As well as we have made the choice and selection of some textbooks as enough representatives of the knowledge to be taught concerning T in each of the 5 countries. An extension would be to look at what happens upstream and downstream of the knowledge to be taught: upstream towards the *scholarly knowledge* and downstream towards the *knowledge taught* and *learned*[13]. This extension would lead us to question the conditions and reasons for the successive transformations of the didactic transposition process and thus enrich this first study.

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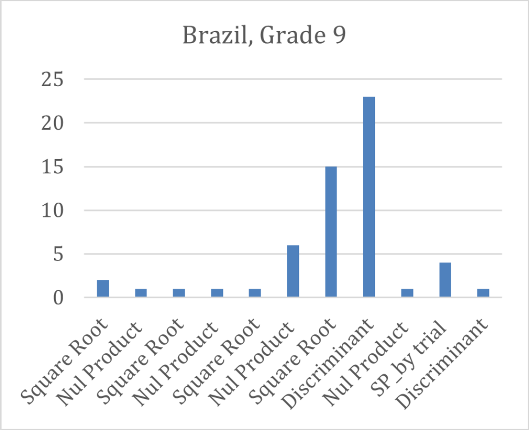
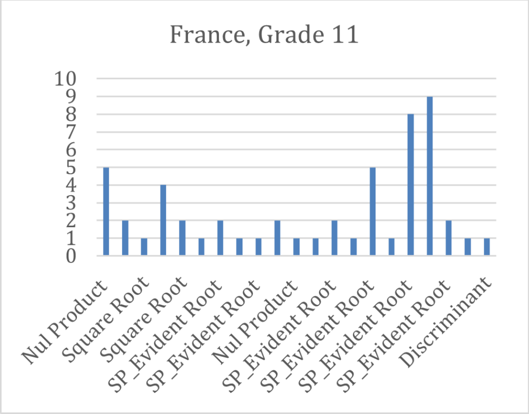
## **Appendix 1** – *A priori* description of the sub-types of tasks $T_i$ and their pragmatic scope

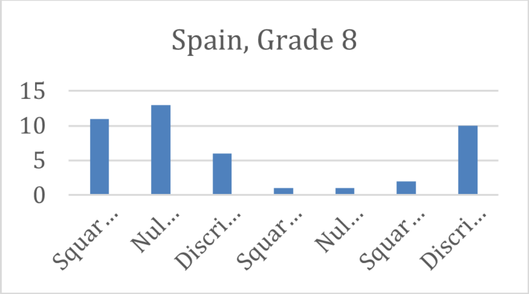
| Sub-type of tasks | Generic expression of $T_i$ | Pragmatic scope |
|---|---|---|
| $T_1$ | $ax^2 + c = 0, ac > 0$ | $\tau_{square\_root}$ |
| $T_2$ | $[nk^2]x^2 = [nk'^2]$ | $\tau_{square\_root}$ |
| $T_3$ | $[nk^2]x^2 - [nk'^2] = 0$ | $\tau_{square\_root}$ |
| $T_4$ | $ax^2 + c = 0; ac < 0$ [C] | $\tau_{square\_root}$ |
| $T_5$ | $ax^2 + c = 0; ac > 0$ [C] | $\tau_{square\_root}$ |
| $T_6$ | $ax^2 + bx = 0$ [C-P] | $\tau_{null\_product}$ |
| $T_7$ | $ax^2 + bx + c = 0$ [C-P] | $\tau_{discriminant}$ |
| $T_8$ | $ax^2 + bx + c = 0; a \pm b \pm c = 0$ or $4a \pm 2b \pm c = 0$ or $9a \pm 3b \pm c = 0$ | $\tau_{discriminant}$ |
| $T_9$ | $[ka^2]x^2 \pm [2kac]x + [kc^2] = 0$ [C-P] | $\tau_{null\_product}$ |
| $T_{10}$ | $k(ax + b)^2 = k'; kk' > 0$ [C-P] | $\tau_{square\_root}$ |
| $T_{11}$ | $k(ax + b)^2 = k'; kk' < 0$ [C-P] | $\tau_{square\_root}$ |
| $T_{12}$ | $[nk^2](ax + b)^2 - [nk'^2] = 0$ [C] | $\tau_{square\_root}$ |
| $T_{13}$ | $[nk^2](ax + b)^2 = [nk'^2]$ | $\tau_{square\_root}$ |
| $T_{14}$ | $[nk^2](ax + b)^2 + [nk'^2] = 0$ [C] | $\tau_{square\_root}$ |
| $T_{15}$ | $[nk^2](ax + b)^2 - [nk'^2](cx + d)^2 = 0$ [C-P] | $\tau_{null\_product}$ |
| $T_{16}$ | $k(ax + b)(cx + d)(\ldots) = 0$ | $\tau_{null\_product}$ |
| $T_{17}$ | $k(ax + b)(cx + d) + k(nax + nb)(ex + f) + \ldots = 0$ [C-P] | $\tau_{null\_product}$ |
| $T_{18}$ | $k(ax + b)(cx + d) + [k'a^2]x^2 - [k'b^2] = 0$ [C-P] | $\tau_{null\_product}$ |
| $T_{19}$ | $P_2(x) = Q(x)$, with apparent factor after partial factorization [P] | $\tau_{null\_product}$ |
| $T_{20}$ | $P_2(x) = Q(x)$, without apparent factorization [P] | $\tau_{discriminant}$ |

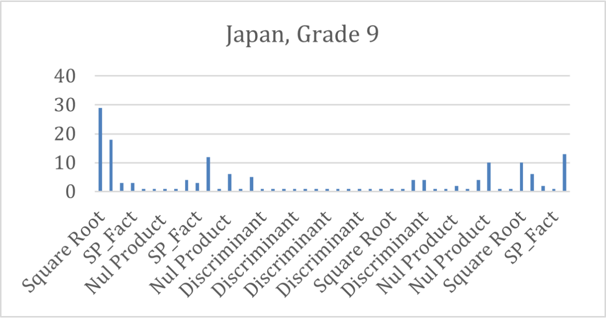
The term [C] means "commutation" of the terms of the equation and [P] "permutation" of the terms with respect to the "=". For example, for T4, the [C] indicates that $ax^2 + c = 0$ et $c + ax^2 = 0$ belong to the same generic expression $ax^2 + c = 0$ with $ac < 0$.

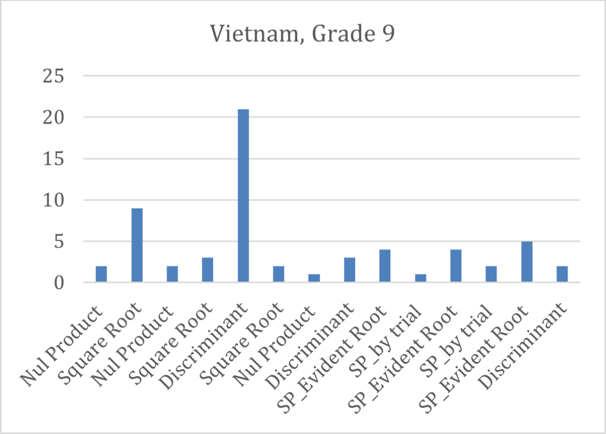
**Appendix 2** – Chronogenesis of the techniques in each country




Japan, Grade 9
40
30
20
10
0
Square Root
SP_Fact
Nul Product
SP_Fact
Nul Product
Discriminant
Discriminant
Discriminant
Discriminant
Square Root
Discriminant
Nul Product
Nul Product
Square Root
SP_Fact


Vietnam, Grade 9
25
20
15
10
5
0
Nul Product
Square Root
Nul Product
Square Root
Discriminant
Square Root
Nul Product
Discriminant
SP_Evident Root
SP_by trial
SP_Evident Root
SP_by trial
SP_Evident Root
Discriminant